MATH 4250/6250 Problem Set 5
Due Friday, February 13

1. A *ruled surface* in $\mathbb{R}^3$ is defined by $\vec{x}(u,v) = \vec{\beta}(u) + v\vec{\delta}(u)$, where $\vec{\beta}(u)$ and $\vec{\delta}(u)$ are curves in $\mathbb{R}^3$. (See Oprea, p. 74-75.)

(a) Give conditions on the curves $\vec{\beta}(u)$ and $\vec{\delta}(u)$ that are necessary and sufficient for $\vec{x}(u,v)$ to be a simple surface. In particular, what conditions on $\vec{\beta}(u)$ and $\vec{\delta}(u)$ give that $\vec{x}_u$ and $\vec{x}_v$ are linearly independent?

(b) Using our graphics software, plot one or more examples ruled surfaces that are regular, and one or more examples that fail to be regular (Assignments, Lab 4).

2. Problems on ruled surfaces from Oprea: Exercises 2.1.24, 2.1.25, 2.1.26.

3. Let S be the subset of $\mathbb{R}^3$ defined by $x = g(u)$ and $y^2 + z^2 = (h(u))^2$, where $\vec{\alpha}(u) = (g(u), h(u), 0)$ is an injective regular plane curve, $a < u < b$, and $h(u) > 0$ for all $u \in (a,b)$ (problem (b) from problem set 4). Prove that the following two simple surfaces are coordinate charts for S, by defining an appropriate coordinate transformation:

$$\vec{x}(s,t) = (g(s), h(s)\cos t, h(s)\sin t),\ a < s < b,\ -\pi < t < \pi$$

$$\vec{y}(u,v) = (g(u), h(u)\cos v, h(u)\sin v),\ a < u < b,\ 0 < v < 2\pi$$

4. Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, and $\mathbb{R}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid z = 0\}$.

(a) If $(s,t,0) \in \mathbb{R}^2$, the line through $(s,t,0)$ and $(0,0,1)$ intersects S^2 at a point other than $(0,0,1)$. Denote this point by $\vec{x}(s,t)$. Compute $\vec{x}(s,t)$ and show that it is a coordinate patch.

(b) Replace the point $(0,0,1)$ in part (a) with the point $(0,0,-1)$, and define $\vec{y}(u,v)$ in the same way. In other words, $\vec{y}(u,v)$ is the point other than $(0,0,-1)$ at which the line between $(u,v,0)$ and $(0,0,-1)$ intersects S^2. Show that $\vec{y}(u,v)$ is a coordinate patch.

(c) Prove that $\vec{x}$ and $\vec{y}$ are coordinate charts for S^2, by defining an appropriate coordinate transformation.