Secant Planes of Space Curves

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015-93

Publication of this manuscript is funded in part by the National Science Foundation.
MSRI wishes to acknowledge support through grant DMS 9022140.

November 1992

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This is joint work with Malcolm Adams, Ted Shifrin and Robert Varley of the University of Georgia[1]. We apply singularity theory to the geometry of 4-dimensional abelian varieties, using the extrinsic geometry of genus 4 canonical curves.

Let C be a smooth algebraic curve in complex projective space $\mathbb{P}^n$, such that C is not contained in a hyperplane. Consider the incidence correspondence

$$\mathcal{X} = \{(p, H) \in C \times \mathbb{P}^{n*} : p \in H\}.$$

We think of $\mathcal{X}$, with its projection to $\mathbb{P}^{n*}$, as the *family of hyperplane sections* of C. The discriminant of this family (the branch locus of the projection) is the dual hypersurface of C.

Suppose C has degree d. We identify the dth symmetric product $C^{(d)}$ with the set of effective divisors of C of degree d. Define $D : \mathbb{P}^{n*} \to C^{(d)}$ by $D(H) = H \cdot C$. The divisor $D(H)$ is the fiber of $\mathcal{X}$ over H. For any divisor $D \in C^{(d)}$, if $D = m_1p_1 + \cdots + m_kp_k$, with $m_i > 0$ and p_i distinct, we say D has *type* $(m_1, \dots, m_k)$.

Proposition. *The family $\mathcal{X}$ is versal if and only if D is transverse to the stratification of $C^{(d)}$ by type.*

Corollary. *If $\mathcal{X}$ is versal then $\{H \in \mathbb{P}^{n*} : D(H)$ has type $(m_1, \dots, m_k)\}$ has codimension $(m_1 - 1) + \cdots + (m_k - 1)$ (i.e., the total order of contact of H with C).*

Is $\mathcal{X}$ versal for generic $C \subset \mathbb{P}^n$ of degree d and genus g? No; for example, Joe Harris has observed that a curve in $\mathbb{P}^4$ of sufficiently high degree and genus violates the conclusion of the corollary, since it lies on a rational normal scroll. Is there such an example for $n = 3$?

Proposition. *The family $\mathcal{X}$ is versal for a generic genus 4 canonical curve.*

To prove the proposition we use that C, which is a curve of degree 6 in $\mathbb{P}^3$, is the intersection of a quadric and a cubic surface. Recently James McKernan has proved that the family $\mathcal{X}$ is versal for a generic canonical curve of arbitrary genus $g > 2$.

Now we look at a closely related map, whose local geometry reflects the global geometry of $\mathcal{X}$. For $C \subset \mathbb{P}^n$ not contained in a hyperplane, consider the rational map $s : C^{(n)} \dashrightarrow \mathbb{P}^{n*}$ which takes a sum of n points $p_1 + \cdots + p_n$ to the span of $p_1, \dots, p_n$. Let Y be the closure

[1]Research supported in part by NSF grant DMS 9106938. McCrory supported in part by NSF grant DMS 9022140 at MSRI.

of the graph of s in $C^{(n)} \times \mathbf{P}^{n*}$. The *secant map* is the projection $S : Y \to \mathbf{P}^{n*}$. The fiber of S over H is the set of effective divisors D of degree n such that $D \leq H{\cdot}C$. For example, if $n = 3$ and $d = 6$, then S has degree $\binom{6}{3} = 20$. If $H{\cdot}C = 2p + 2q + 2r$ (H is a tritangent plane), then the fiber of S over H has one point of multiplicity 8: $p+q+r$, and six points of multiplicity 2: $2p+q$, $2p+r$, $2q+p$, $2q+r$, $2r+p$, $2r+q$. The branch locus of S is the dual hypersurface of C.

The following intrinsic description of the secant map of a canonical curve is our motivation for studying it. For a smooth genus g curve C, consider the Jacobian variety

$$J(C) = \frac{H^0(C,K)^*}{H_1(C,\mathbf{Z})} = \mathbf{C}^g/L,$$

the duals of holomorphic differentials, mod periods (a g-dimensional complex vector space modulo a lattice). Let $\Theta \subset J(C)$ be the theta divisor.

Torelli's Theorem. *The pair $(J(C), \Theta)$ determines the curve C.*

Andreotti's beautiful proof goes roughly as follows. Consider the *Gauss map*

$$\gamma : \Theta \to \mathbf{P}(T_0 J(C))^*,$$

which assigns to a point of Θ its tangent plane translated to the origin of $J(C)$. The branch locus B of γ is the dual hypersurface of the canonical embedding $C \subset \mathbf{P}^{g-1}$, so $C = B^*$, q.e.d.!

This description of the branch locus of γ is a consequence of the following theorem of Riemann. The canonical map takes $p \in C$ to $(\omega_1(p) : \cdots : \omega_g(p))$, where $\omega_1, \ldots, \omega_g$ is a basis for the holomorphic differentials on C. The canonical map is an embedding if C is nonhyperelliptic. Let $\vec{\omega} = (\omega_1, \ldots, \omega_g)$.

Theorem. *Θ is the image of the Abel-Jacobi map $\mu : C^{(g-1)} \to J(C)$,*

$$\mu(p_1 + \cdots + p_{g-1}) = \int_{p_o}^{p_1} \vec{\omega} + \cdots + \int_{p_o}^{p_{g-1}} \vec{\omega} \ \ \textit{(mod periods).}$$

Since the theta divisor Θ may have singularities, the Gauss map γ is actually defined on the Nash blowup $\widehat{\Theta}$. The Abel-Jacobi map lifts to a finite map $\hat{\mu} : Y \to \widehat{\Theta}$.

Corollary. *If C is nonhyperelliptic, then $\gamma \circ \hat{\mu} = S$.*

In other words, the Gauss map of the theta divisor corresponds to the secant map of the canonical embedding.

The Gauss map γ has interesting geometric structure; γ is the $\mathbf{C}^*$-quotient of a *conic Lagrangian map* Γ *with* $\mathbf{Z}/2$*-symmetry*:

$$\begin{array}{ccc} N^*(\Theta) & \xrightarrow{\Gamma} & T_0^*J(C) - 0 \\ \downarrow & & \downarrow \\ \widehat{\Theta} & \xrightarrow{\gamma} & \mathbf{P}T_0^*J(C) \end{array}$$

The conic $\mathbf{Z}/2$-Lagrangian map Γ is the *homogeneous Gauss map*. Here $N^*(\Theta) \subset T^*J(C) - 0$ is the conormal variety of Θ, and Γ is the restriction of the projection to the fiber over $0 \in J(C)$. The symmetry of Γ is induced by the (-1)-map of $J(C)$. The corresponding symmetry of S is given by $D \mapsto H \cdot C - D$, for a divisor D of degree $g-1$ such that $D \leq H \cdot C$. (The canonical embedding $C \subset \mathbf{P}^{g-1}$ has degree $2g-2$.)

If C is a nonhyperelliptic genus 4 curve, then $\widehat{\Theta}$ is smooth and $\hat{\mu}$ is an isomorphism; thus we can identify the Gauss map γ with the secant map S. Using the geometry of the family of hyperplane sections of the canonical embedding, we can show that if C is a generic genus 4 curve, then Γ has local singularities of types A_1, A_2, A_3, A_4, D_4, D_6, E_6, and $P_8(\mathbf{Z}/2) = \widetilde{E}_6(\mathbf{Z}/2)$. Some of these singularities are not generic conic $\mathbf{Z}/2$-Lagrangian singularities; e.g. $\dim D_4 > 0$, $D_6 \neq \emptyset$, and $E_6 \neq \emptyset$. In terms of the geometry of the secant map S, a D_4 singularity corresponds to $p+q+r \subset 2p+2q+r+s$, a bitangent plane, and a P_8 singularity corresponds to $p+q+r \subset 2p+2q+2r$, a tritangent plane. The other singularity types have similar interpretations.

We apply our knowledge of the singularities of the Gauss map of a Jacobian theta divisor to the study of generic principally polarized abelian varieties. Our philosophy is that the branch locus B of the Gauss map γ is the basic projective invariant of a principally polarized abelian variety (A, Θ). We ask whether there is a "Torelli" theorem: Are two principally polarized abelian varieties isomorphic if and only if their Gauss maps have isomorphic branch loci? We also ask whether there is a "Schottky" theorem characterizing the moduli space $\mathcal{J}_g$ of Jacobian varieties in the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties: Is $\mathcal{J}_g$ an irreducible component of the locus of $\mathcal{A}_g$ consisting of (A, Θ) such that the dual of B is a curve? The following theorem gives a classification of the singularities of B in dimension 4.

Theorem. *For a generic principally polarized abelian variety* (A, Θ) *of dimension 4, the homogeneous Gauss map* Γ *has generic conic* $\mathbf{Z}/2$*-Lagrangian singularities.*

We prove this result by deformation of the theta function ϑ of a Jacobian, using that ϑ satisfies the heat equation

$$\frac{\partial \vartheta}{\partial \Omega_{ij}} \sim \frac{\partial^2 \vartheta}{\partial z_i \partial z_j}.$$

One case of the proof involves the P_8 singularities, which occur at the 120 fixed points of the (-1)-involution of Θ (the odd theta characteristics). We show that, for generic (A, Θ), the Gauss map γ has rank less than 2 only at the 120 fixed points. At each fixed point, the branch locus of γ is isomorphic to the branch locus of the gradient of $x^3 + y^3 + z^3 + axyz$, where a is a complex parameter (a modulus of the P_8 singularity) which is not identically zero as (A, Θ) varies, but which vanishes for Jacobians.

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