PROFILES OF SURFACES

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A profile, or apparent contour, of a surface in Euclidean space is the set of critical values of a projection of the surface to a plane. A line orthogonal to this projection plane passes through a point of the profile if and only if it is tangent to the surface. For almost all projection planes, the profile of a smooth surface is a smooth curve with isolated cusps. This corresponds to Whitney's classification of the singularities of stable mappings of surfaces [6]. As the plane turns, pairs of cusps of the profile can be created or destroyed, as when a torus of revolution is projected to planes containing a fixed line orthogonal to its axis of symmetry. The simplest ways that cusps can be born or die are the swallowtail, lip and beaks transitions of Thom [4].

A cusp of a profile occurs when a surface is projected along an asymptotic direction. Koenderink and van Doorn [3] observed that the inflections of the profile also have geometric significance: they correspond to parabolic points of the surface. Koenderink and van Doorn also listed the simplest ways that inflections of the profile can be born and die as the projection plane turns.

In this note I describe a classification of the local geometry of the family of all profiles of a generic surface in Euclidean space. This classification refines Gaffney and Ruas' unpublished classification of the singularities which occur in the family of projections to planes of a generic surface. (Their classification allows smooth changes of coordinates, which can erase the inflections of profiles.) At the same time I extend Koenderink and van Doorn's list of visual transitions to include those phenomena which occur in general only for isolated viewing directions.

This note is a summary of joint work with Thomas Banchoff and Terence Gaffney at Brown University, partially supported by the National Science Foundation. It is a sequel to our study of the classical Gauss map [1].

Similar work has been done by Y. L. Kergosien [7]

1. Projections to planes

Let $X: M^2 \to R^3$ be a smooth immersion of the surface M in Euclidean space. For each unit vector V in R^3, let ξ^V be the plane through the origin perpendicular to V, and let

$$\Pi^{XV}: M^2 \to \xi^V$$
$$\Pi^{XV}(P) = X(P) - (X(P)\cdot V)V,$$

the composition of X with orthogonal projection to ξ^V. The profile C(X,V) of the immersion X in the direction V is the set of critical values of the map Π^{XV}:

$$C(X,V) = \Pi^{XV}(\{P \in M | \text{rank } d\,(\Pi^{XV})_P = 1\}).$$

(Note that rank $d(\Pi^{XV})_P \geq 1$ for all $P \in M$, since X has rank 2, and orthogonal projection to ξ^V has Kernel rank 1 everywhere.)

Let TS^2 be the tangent bundle of the unit sphere S^2 in R^3. Identifying ξ^V with the tangent plane to S^2 at V gives a family of maps parametrized by S^2:

$$\Pi^X: S^2 \times M^2 \to TS^2$$

(A)

$$\Pi^X(V,P) = (V,\Pi^{XV}(P))$$

The critical set of this family is the set of pairs (V,P) such that V is parallel to the tangent plane of X at P, so the critical set can be identified with the unit tangent circle bundle of X.

<u>Theorem 1</u> (Gaffney and Ruas 1977 [1]). cf. Arnold [8].

Let M^2 be a smooth surface. For an open dense subset A(M) of the space of immersions $X:M^2 \to R^3$, the family Π^X is versal.

The topology on the space of immersions is the Whitney C^∞ topology.

Gaffney and Ruas' proof is based on an explicit classification of all rank one finitely determined germs $R^2, 0 \to R^2, 0$ of codimension 2 or less. As a corollary of the proof, they obtained the following classification. For $X \in A(M)$, the germ of Π^{XV} at $P \in M$ and $Q \in \xi^V$ is equivalent, under C^∞ coordinate changes in the source and target, to one of the following ten germs:

germ $(x,y) \mapsto$	name	codimension
1. (x,y)	ordinary	
2. (x,y^2)	fold	0
3. (x,y^3+xy)	cusp	0
4. (x,y^4+xy)	swallowtail	1
5. $(x,y^5+xy+xy^3)$	butterfly	2
6. $(x,y^5+xy-xy^3)$	butterfly	2
7. (x,y^3+x^2y)	lips	1
8. (x,y^3-x^2y)	beaks	1
9. (x,y^3+x^3y)	goose	2
10. $(x,y^4+x^2y+xy^2)$	gulls	2

Gaffney and Ruas also investigated the geometric properties of these germs. If Π^{XV} has a fold at P, then V is parallel to the tangent plane of X at P. If it has a cusp, then P is hyperbolic, and V is parallel to an asymptotic direction of X at P. If it has a swallowtail then not only is

P hyperbolic, with V parallel to an asymptotic direction of X at P, but also the asymptotic curve through P in this direction has an inflection at P. If Π^{XV} has a butterfly at P, then P is hyperbolic, V is parallel to an asymptotic direction at P, and the asymptotic curve through P in this direction has an undulation at P. (I do not know how to distinguish the two butterflies geometrically.) At a lips, beaks, or goose singularity, P is a parabolic point of X, and V is parallel to the unique asymptotic direction of X at P. (These three singularities have distinct profiles.) At a gulls singularity, P is parabolic, V is parallel to the unique asymptotic direction at P, and this direction is tangent to the parabolic curve of X at P.

The singularities of the projection Π^{XV} reflect the order of contact of the immersion X with the pencil of lines parallel to V. At a fold of Π^{XV}, a line parallel to V has first order contact (tangency) with X. A cusp reflects second order contact, a swallowtail third order contact, and a butterfly fourth order contact [1].

2. Projections to lines

To classify the inflection properties of the profiles of a surface I shall also consider the contact of the surface with planes, which is reflected by the singularities of projections to lines. Let $X:M^2 \to R^3$ be a smooth immersion. For each unit vector W in R^3 let ξ_W be the line spanned by W, and consider

$$\Pi_{XW} : M^2 \to \xi_W$$
$$\Pi_{XW}(P) = (X(P)\cdot W)W,$$

the composition of X with orthogonal projection to ξ_W. If NS^2 is the normal bundle of S^2, this gives a family of real-valued functions on M parametrized by S^2:

$$\Pi_X : S^2 \times M^2 \to NS^2 = S^2 \times R$$

(B)

$$\Pi_X(W,P) = (W,\Pi_{XW}(P))$$

The critical set Σ of this family is just the unit normal bundle of X, which is the oriented double covering space of M. The projection of Σ to the parameter space S^2 (the catastrophe map of the family (B)) is the Gauss map of the immersion X (cf. [1]).

Theorem 2 (Looijenga 1974 [5]).

Let M^2 be a smooth surface. For an open dense subset B(M) of the space of immersions $X:M^2 \to R^3$, the family Π_X is versal.

For $X \in B(M)$, the germ of Π_{XW} at $P \in M$ and $Q \in \xi_W$ is equivalent, under C^∞ coordinate changes in source and target, to one of the following six germs:

	<u>germ</u> $(x,y) \mapsto$	<u>codimension</u>	<u>Morse index</u>	<u>zero level curve</u>
a.	y			line
b.	$x^2 + y^2$	0	+1	point
c.	$x^2 - y^2$	0	-1	node
d.	$x^2 + y^3$	1	0	cusp
e.	$x^2 + y^4$	2	+1	point
f.	$x^2 - y^4$	2	-1	tacnode

If Π_{XW} has a singularity of type b (an <u>extremum</u>) at P, then P is an elliptic point of the immersion X. If Π_{XW} has a singularity of type c (a <u>saddle</u>) at P, then P is hyperbolic. If Π_{XW} has a singularity of type d, e, or f at P, then P is parabolic. The singularities of Π_{XW} reflect the contact of the immersion X with the family of planes perpendicular to W. If P is a critical point of Π_{XW}, then the plane through P perpendicular to W is tangent to X at P, i.e. it has first order contact with X at P. At points of type d, e, or f, this plane has stationary contact with X at P. If Π_{XW} has a singularity of type d at P, then there is a unit vector V near W so that the family of planes perpendicular to V is tangent to X at two points arbitrarily near P. If Π_{XW} has a singularity of type e or f at P, then there is a vector V near W so that the family of planes perpendicular to V is tangent to X at three points arbitrarily near P. Dually, if Π_{XW} has a singularity of codimension n at P, then there is a vector V near W so that Π_{XV} has n+1 nondegenerate critical points (i.e.

extrema or saddles) arbitrarily near P. (In other words, Π_{XW} has Milnor number n+1 at P.) The Morse index of the singularity is then the number of resulting extrema minus the number of resulting saddles.

3. A hybrid family.

Now I shall construct a family of maps which will be used to relate the singularities of projections to planes and the singularities of projections to lines.

For each unit vector V in R^3, let $P(V)$ be the set of unit vectors perpendicular to V, and let $L(V)$ be the set of pairs (W,Q) with $W \in P(V)$ and $Q \in \xi_W$, the line spanned by W. Consider the map

$$\hat{\Pi}_{XV}:P(V) \times M \to L(V)$$

$$\hat{\Pi}_{XV}(W,P) = (W,\Pi_{XW}(P)) \quad .$$

Now let $P(S^2)$ be the set of pairs (V,W) of perpendicular unit vectors, and let $L(S^2)$ be the set of triples (V,W,Q) with $(V,W) \in P(S^2)$ and $Q \in \xi_W$. Projection to the first factor makes $P(S^2)$ a circle bundle over S^2, isomorphic with the tangent circle bundle of S^2. Projection to the first two factors makes $L(S^2)$ a line bundle over $P(S^2)$, isomorphic with the canonical line bundle. We obtain a family of maps parametrized by S^2:

$$\hat{\Pi}_X:P(S^2) \times M^2 \to L(S^2)$$

(C)

$$\hat{\Pi}_X((V,W),P) = (V,W,\Pi_{XW}(P)) \quad .$$

Using theorem 2, and the relation of the family (C) to the family (B), we obtain the following result.

<u>Theorem 3</u>. Let M^2 be a smooth surface. For $X:M^2 \to R^3$ an immersion in the open dense subspace $B(M)$ of the space of immersions of X in R^3, the family $\hat{\Pi}_X$ is versal.

According to Gaffney and Ruas, if the family (A) of projections to planes is versal, then so is the family (B) of projections to lines. Therefore all three families (A), (B), and (C) are versal for $X \in A(M)$.

Now I shall show that the singularities of the family (C) are dual to those of the family (A). Recall that the profile $C(X,V)$ of the immersion $X:M^2 \to R^3$ on the plane ξ^V is the set of critical values of the map $\Pi^{XV}:M^2 \times \xi^V$. Let $\hat{C}(X,V)$ be the set of critical values of the map $\hat{\Pi}_{XV}:P(V) \times M^2 \to L(V)$ (a map from a 3-manifold to a 2-manifold). Let $p:L(V) \to \xi^V$ be projection on the second factor $(p(W,Q) = Q)$. Note that $p|\{(W,Q) \in L|\ Q \neq 0\}$ is a submersion.

<u>Proposition</u>. If $X \in A(M)$ then $p(\hat{C}(X,V))$ is the pedal curve from the origin of the profile $C(X,V)$.

<u>Proof</u>: If $X \in A(M)$ then $C(X,V)$ is a smooth curve with isolated singular points. Each critical point P of Π^{XV} determines a tangent line T_P to $C(X,V)$ at $\Pi^{XV}(P)$, even if $\Pi^{XV}(P)$ is a singular point of $C(X,V)$: let $T_P = d\Pi^{XV}(TM_P)$, the projection of the tangent plane of M at P to the plane ξ^V. So $C(X,V)$ has a well-defined pedal curve, the locus of points $Q \in \xi^V$ such that Q is the foot of the perpendicular from the origin to the line T_P for some critical point P of Π^{XV}.

If $Q \in \xi^V$ and W is a unit vector in the direction of Q, then Q is the foot of the perpendicular from the origin to the line T_P if and only if P is a critical point of Π_{XW} with critical value Q. This is equivalent to the statement that P is a critical point of $\hat{\Pi}_{XV}$ with critical value (W,Q) . This completes the proof.

Remark. This proof shows that the dual curve of a profile of a surface is a plane section of the dual surface (cf. [2]).

The singularities of the pedal curve $\hat{C}$ of a plane curve C are dual to the singularities of C . In particular, cusps of $\hat{C}$ correspond to inflections of C , and vice versa.

The classification (a-f) of the singularities of the family (B) gives a corresponding classification of the singularities of the family (C) , which in turn gives a classification of the dual singularities of the family of profiles of an immersion.

4. Classification of profiles.

The local geometry of the family of profiles of the immersion $X:M^2 \to R^3$ is determined by the family of pairs of germs $(\mathrm{II}^{XV}, \mathrm{II}_{XW})$ with V and W perpendicular unit vectors. For X in the generic set $A(M)$ the following ~~fifteen~~ sixteen pairs of germs can occur. The immersion $X(x,y) = (x,y,f(x,y))$ is abbreviated to $z = f(x,y)$. In each example $P = (0,0)$ and $W = (0,0,1)$.

1a <u>ordinary</u>. The profile is empty.

2b <u>elliptic fold</u>. The profile is convex.
e.g. $z = x^2 + y^2$, $V = (1,0,0)$.

2c <u>hyperbolic fold</u>. The profile is concave.
e.g. $z = y^3 + xy + x^2$, $V = (\sqrt{2}/2, \sqrt{2}/2, 0)$.

2d <u>parabolic fold</u>. The profile has an inflection.
e.g. $z = y^3 + x^2y + x^2$, $V = (1,0,0)$. This is a cusp of the pedal curve of the profile.

2e <u>elliptic undulation</u> (or hyperbolic intrusion [3]). Two inflections of the profile are created or destroyed. The profile is convex. e.g. $z = y^4 + x^2y + xy^2 + x^2$, $V = (1,0,0)$. This is dual to a swallowtail of the pedal curve.

2f <u>hyperbolic undulation</u> (or elliptic intrusion [3]). Two inflections of the profile are created or destroyed. The profile is concave. e.g. $z = y^4 + x^2y + xy^2 - x^2$, $V = (1,0,0)$. This is dual to a swallowtail of the pedal curve.

3c <u>cusp</u>. e.g. $z = y^3 + xy + x^2$, $V = (0,1,0)$. This is dual to an inflection of the pedal curve.

4c <u>swallowtail</u>. Two cusps sharing one branch are created or destroyed. e.g. $z = y^4 + xy + x^2$, $V = (0,1,0)$. This is dual to an undulation of the pedal curve.

5c <u>butterfly</u>. Two swallowtails coalesce, sharing a cusp. e.g. $z = y^5 + xy + xy^3 + x^2$, $V = (0,1,0)$. This is dual to a point of the pedal curve where it has fourth order contact with its tangent line.

6c <u>butterfly</u>. Same description as (5c), but not an equivalent singularity. e.g. $z = y^5 + xy - xy^3$, $V = (0,1,0)$.

7d <u>lips</u>. Two cusps sharing both branches are created or destroyed, and two inflections are created or destroyed at the same time. e.g. $z = y^3 + x^2y + x^2$, $V = (0,1,0)$. This is dual to a lips singularity of the pedal curve.

8d <u>beaks</u>. Two cusps sharing neither branch are created or destroyed, and two inflections are created or destroyed at the same time. e.g. $z = y^3 - x^2y + x^2$, $V = (0,1,0)$. This is dual to a beaks singularity of the pedal curve.

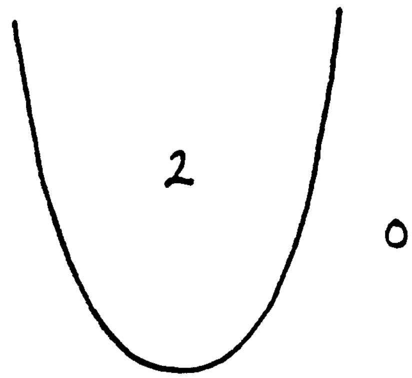
9d <u>goose</u>. This is the transition between lips and beaks. At the instant of transition, the profile has a rhamphoid cusp. e.g. $z = y^3 + x^3y + x^2$, $V = (0,1,0)$.

10e elliptic gulls. This is the coalescence of beaks, swallowtail, and elliptic undulation. e.g. $z = y^4 + x^2y + xy^2 + x^2$, $V = (0,1,0)$.

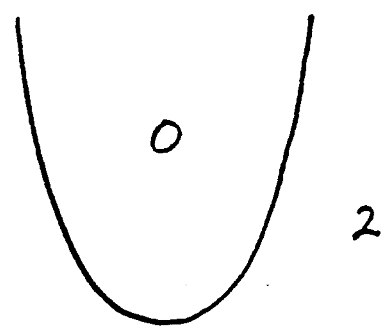
10f hyperbolic gulls. This is the coalescence of beaks, swallowtail, and hyperbolic undulation. e.g. $z = y^4 + x^2y + xy^2 - x^2$, $V = (0,1,0)$.

10f hyperbolic gulls second type.

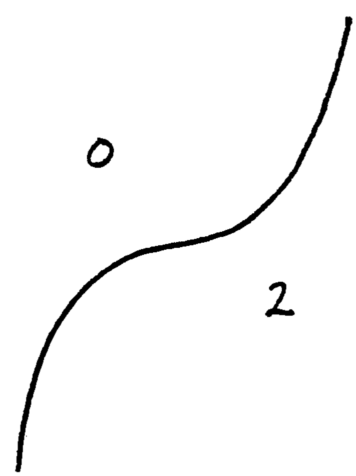
In the following illustrations of the ~~fourteen~~ fifteen nonempty types of profiles, the numbers tell how many times the surface covers each region of the projection plane. The codimension 2 singularities (5c, 6c, 9d, 10e, 10f) are illustrated by one or more nearby projections. Note that there is no double tangent line on the beaks profile, since this would correspond to a double point of the pedal curve, which also has a beaks singularity. This corrects the "bell-split" and "bowl split" figures of [3, p.56]. (This restriction makes the cusps of the beaks singularity appear to be rhamphoid. But the only rhamphoid cusp in a profile of a generic surface occurs precisely at the goose singularity.)



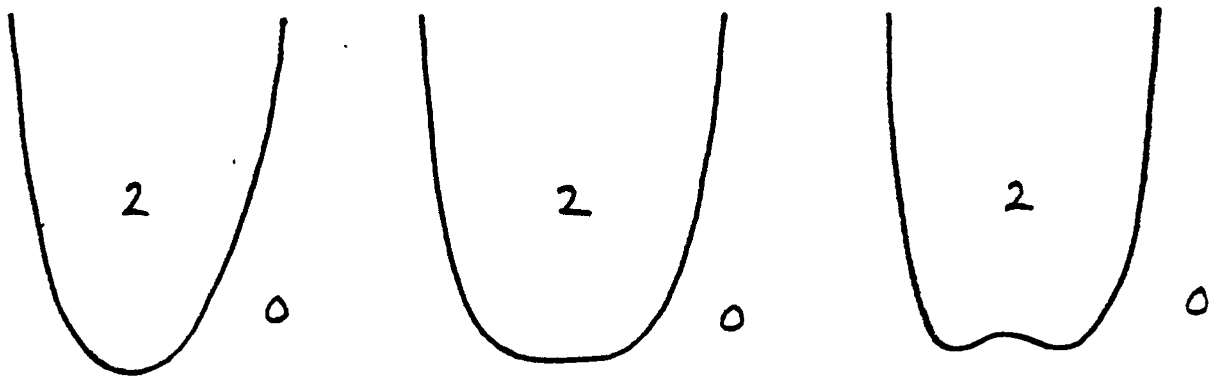
2b elliptic fold



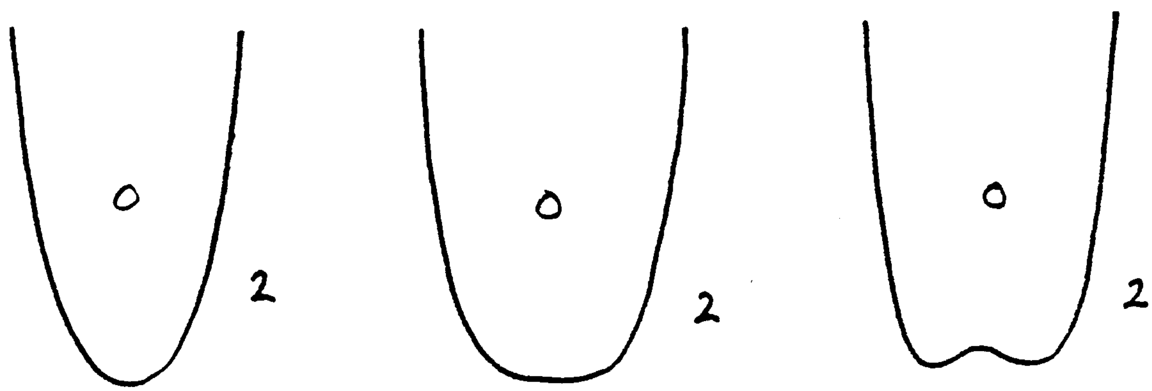
2c hyperbolic fold



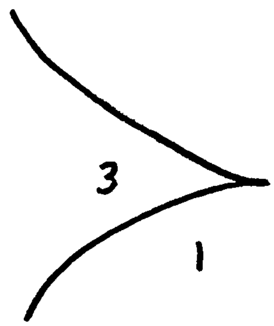
2d parabolic fold



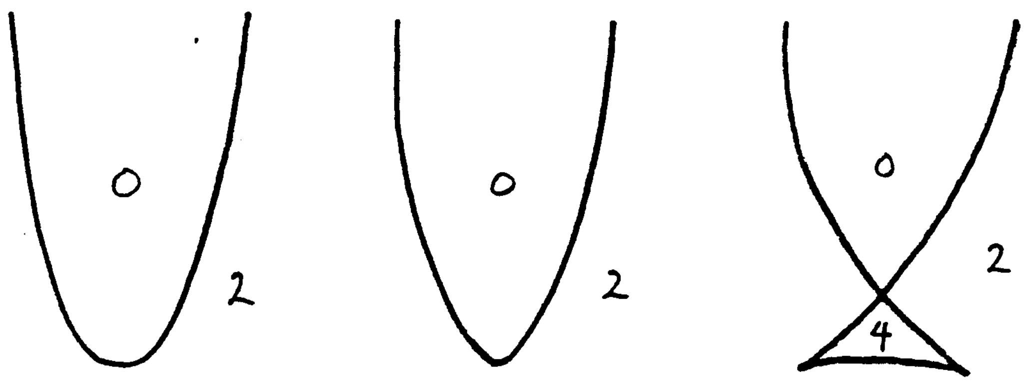
2e elliptic undulation



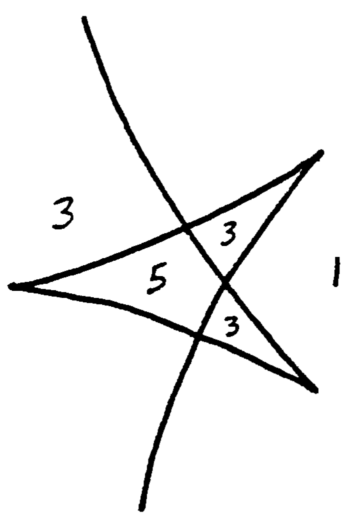
2f hyperbolic undulation



3c cusp

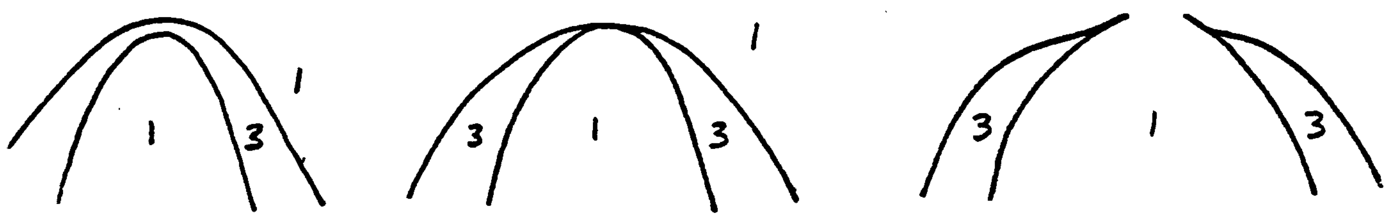


4c swallowtail

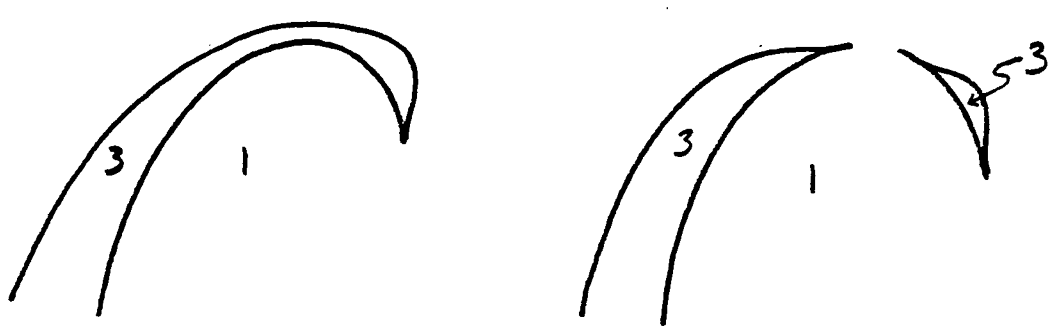


5c, 6c butterfly

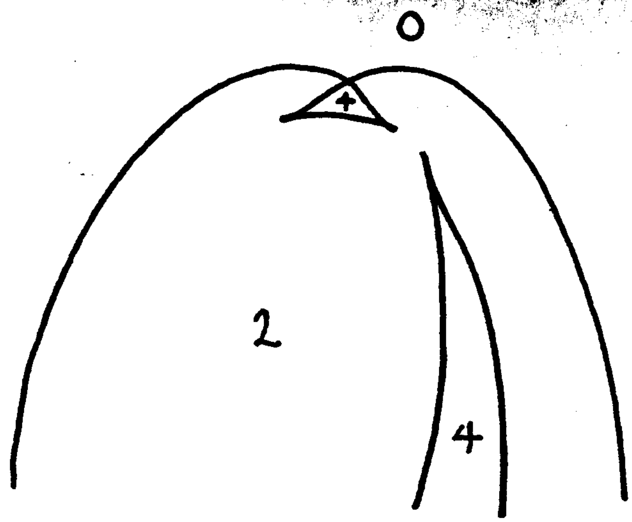
7d lips



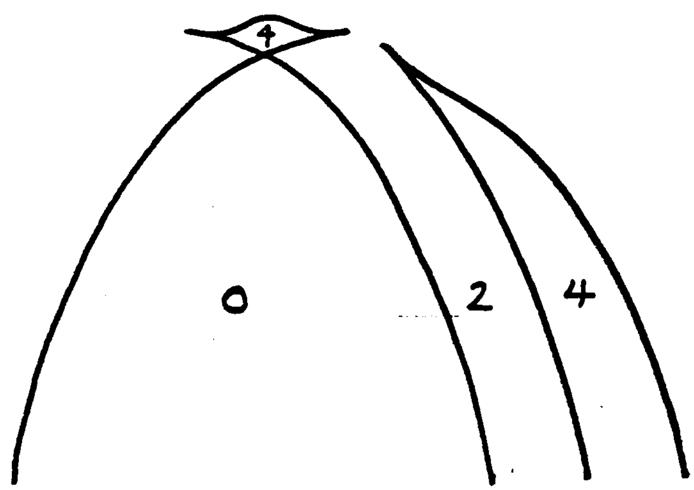
8d beaks



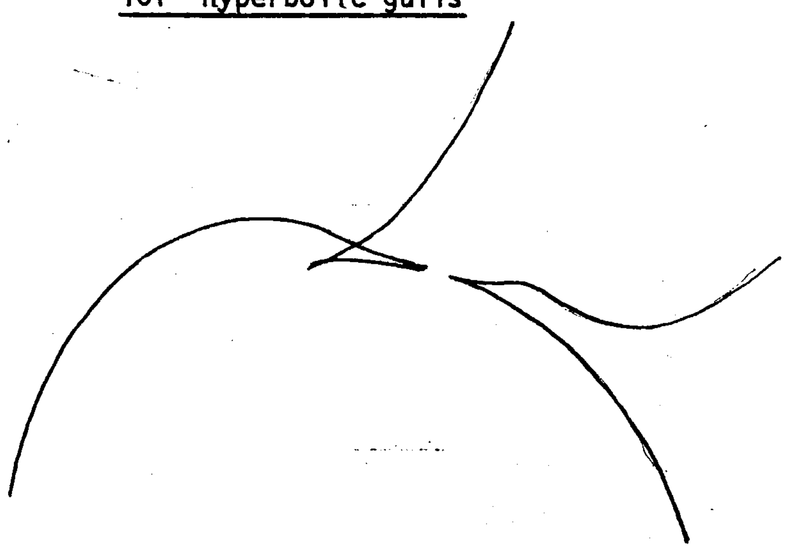
9d goose



10e elliptic gulls



10f hyperbolic gulls



10f hyperbolic gulls, second type

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