

MATH 2200
TEST 2

Name: _____ Key _____

Problem 1: (30 points) A rectangle is inscribed in a semicircle of radius 10 as shown below. What is the maximum possible area of such a rectangle?

Solution: Sorry I couldn't reproduce the picture. I'll compensate with two solutions.

Solution 1: Let x be the height of the rectangle and y be half the width of the rectangle. Then $x^2 + y^2 = 100$ and the area of the rectangle is $2xy$. So $A(x) = 2x\sqrt{100 - x^2}$, where x is in the closed interval $[0, 10]$. Then we get

$$A'(x) = 2\sqrt{100 - x^2} + 2x \frac{1}{2\sqrt{100 - x^2}}(-2x)$$

This only has a problem in our interval at 10, which is already on our list anyway. Setting $A'(x) = 0$, we get

$$\begin{aligned} 0 &= 2\sqrt{100 - x^2} - \frac{2x^2}{\sqrt{100 - x^2}} \\ \frac{2x^2}{\sqrt{100 - x^2}} &= 2\sqrt{100 - x^2} \\ 2x^2 &= 2(100 - x^2) \\ 2x^2 &= 200 - 2x^2 \\ 4x^2 &= 200 \end{aligned}$$

So $x = \sqrt{50}$ is the critical point inside the interval. Since $A(0) = A(10) = 0$, $A(\sqrt{50}) = 100$ is the max area.

Solution 2: Let θ be the angle that the radius up to the top right corner makes with the horizontal. Then the height of the rectangle is $10 \sin \theta$ and half its width is $10 \cos \theta$, so its area is $A(\theta) = 200 \sin \theta \cos \theta$, where θ is in the closed interval $[0, \pi/2]$. Setting $A'(\theta) = 0$, we get

$$0 = 200 \cos \theta \cos \theta + 200 \sin \theta (-\sin \theta) = 200(\cos^2 \theta - \sin^2 \theta).$$

So $\cos^2 \theta = \sin^2 \theta$, and the only solution to this equation in $[0, \pi/2]$ is $\theta = \pi/4$. Since $A(0) = A(\pi/2) = 0$, $A(\pi/4) = 100$ is the max area.

(Remark: notice that $A(\theta) = 100 \sin(2\theta)$. Clearly this is maximized when $\sin(2\theta) = 1$. No calculus necessary there, if you know the double-angle formula.)

Problem 2: (20 points) Let $f(x) = \sqrt{x + \sqrt{x}}$. Find $f'(x)$.

Solution: Just remember the Chain Rule:

$$f'(x) = \frac{1}{2\sqrt{x + \sqrt{x}}}\left(1 + \frac{1}{2\sqrt{x}}\right).$$

No simplification required.

Problem 3: (25 points) A plane is flying horizontally over an observer at an altitude of 3 km, at a constant speed of 10 km/min.

(a) (5 points) Let x be the distance from the observer to the point on the ground directly under the plane. Give a formula for the distance $D(x)$ from the observer to the plane.

Solution: By the Pythagorean Theorem, $D(x) = \sqrt{9 + x^2}$.

(b) (20 points) When $x = 4$ km, how fast is the distance $D(x)$ from the observer to the plane increasing? (Remember your units.)

Solution: We want $\frac{dD}{dt} = \frac{dD}{dx} \frac{dx}{dt} = \frac{dD}{dx} \cdot 10$ by the Chain Rule. And

$$\frac{dD}{dx} = \frac{1}{2\sqrt{9+x^2}}(2x),$$

and we want to plug in $x = 4$, so we get $\frac{dD}{dx} = \frac{1}{10}(8) = \frac{4}{5}$.

Then we get $\frac{dD}{dt} = \frac{4}{5} \cdot 10 = 8$ km/min.

Problem 4: Let $f(\theta) = \sin \theta + \cos \theta$.

(a) (15 points) Find *all* c in the interval $[0, 2\pi]$ such that $f'(c) = 0$.

Solution: We have that $f'(\theta) = \cos \theta - \sin \theta$, so setting it equal to 0 yields $\cos \theta = \sin \theta$. There are two solutions to this equation in $[0, 2\pi]$, namely $\pi/4$ and $5\pi/4$.

(b) (10 points) Find the maximum and minimum values of f on $[0, 2\pi]$.

Solution: Make your list:

$$\begin{aligned}f(0) &= 1 \\f(\pi/4) &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \\f(5\pi/4) &= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2} \\f(2\pi) &= 1\end{aligned}$$

So the maximum value is $\sqrt{2}$ and the minimum value is $-\sqrt{2}$.