MATH 2200
TEST 3

Name: Key

Problem 1: (25 points) A gas balloon is being filled at the rate of 100π cm^3 of gas per second. At what rate is the radius of the balloon increasing when its radius is 10 cm? (The volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.)

Solution: We have $V = \frac{4}{3}\pi r^3$, and we want $\frac{dr}{dt}$ when $r = 10$, given that $\frac{dV}{dt} = 100\pi$. So

$$\begin{aligned}\frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt}\\ 100\pi &= 4\pi (10)^2 \frac{dr}{dt}\\ \frac{100\pi}{4\pi(10)^2} &= \frac{dr}{dt}\\ \frac{1}{4} &= \frac{dr}{dt}\end{aligned}$$

So the answer is 1/4 cm/sec.

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Problem 2: (25 points) Estimate $\sqrt[3]{25}$. You can use linear approximations or differentials– your choice.

Solution: Let $f(x) = \sqrt[3]{x} = x^{1/3}$, so $f'(x) = \dfrac{1}{3}x^{-2/3}$. If we use differentials, we get

$$\begin{aligned}\Delta f &\approx f'(x)\Delta x \\ &= f'(27)(25-27) \\ &= \frac{1}{3}(27)^{-2/3}(-2) \\ &= -\frac{2}{27},\end{aligned}$$

so $f(25) \approx f(27) + \Delta f = 3 - \dfrac{2}{27} = \dfrac{79}{27}$.

If we use linear approximations with $a = 27$ and $x = 25$, we get

$$\begin{aligned}f(x) \approx L(x) &= f(a) + f'(a)(x-a) \\ &= f(27) + f'(27)(25-27) \\ &= 3 - \frac{2}{27}\end{aligned}$$

as above.

Problem 3: (25 points) A box with a square base and an open top is to have volume 4 ft^3. What is the minimum possible surface area of the box?

Solution: Let x be the length of the base, and y be the height of the box. Then $x^2y = 4$, so $y = 4/x^2$. Now the surface area equals $4xy + x^2$, which is

$$A(x) = 4x\left(\frac{4}{x^2}\right) + x^2 = \frac{16}{x} + x^2.$$

This function makes sense on the open interval $(0, \infty)$. To find the critical points, set $A' = 0$ to get

$$\begin{aligned} 0 &= -\frac{16}{x^2} + 2x \\ \frac{16}{x^2} &= 2x \\ 16 &= 2x^3 \\ 8 &= x^3 \\ 2 &= x \end{aligned}$$

Now make your number line with 2 marked. We have $A'(1) = -16 + 2 = -14$ and $A'(4) = -1 + 8 = 7$. So the derivative is negative to the left of 2 and positive to the right, so 2 is a global minimum by the First Derivative Test (because there's only one critical point).

Finally, the answer is $A(2) = \frac{16}{2} + 2^2 = 12$ ft^2.

Problem 4: Let C be the curve given by the equation $x^2 - xy + y^2 = 3$.

(a) (10 points) Find a formula for $\frac{dy}{dx}$. (Your answer should be in terms of x and y.)

Solution: We have

$$2x - \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$

$$2x - y = \frac{dy}{dx}(x - 2y)$$

$$\frac{2x - y}{x - 2y} = \frac{dy}{dx}$$

(b) (5 points) Write the equation of the line tangent to C at the point $(-1, 1)$.

Solution: Plugging into the formula in part (a), we get $\frac{dy}{dx} = \frac{2(-1) - 1}{-1 - 2(1)} = 1$, so we get $y - 1 = 1(x + 1)$, so $y = x + 2$ is our answer.

(c) (10 points) At which points on C is the tangent line horizontal? At which points on C is the tangent line vertical?

Solution: To get the horizontal tangent line, set the numerator of $\frac{dy}{dx}$ equal to 0, so $2x - y = 0$, so $y = 2x$. Now plug into the equation of the curve: $x^2 - x(2x) + (2x)^2 = 3$, so $x^2 - 2x^2 + 4x^2 = 3$, so $3x^2 = 3$, so $x^2 = 1$, so $x = \pm 1$. Since $y = 2x$, we get the two points $(1, 2)$ and $(-1, -2)$.

To get the vertical tangent line, set the denominator of $\frac{dy}{dx}$ equal to 0, so $x - 2y = 0$, so $x = 2y$. Now plug into the equation of the curve: $(2y)^2 - (2y)y + y^2 = 3$, so just as above we get $y = \pm 1$. Since $x = 2y$, we get the two points $(2, 1)$ and $(-2, -1)$.