MATH 4400
HOMEWORK 4
DUE 2/19/08

Instructions: Math 6400 students should do all the problems. Math 4400 students should do problems 1,2,4,5,6; in addition, they should do three of the five parts of problem 3.

Problem 1: Let f be a multiplicative arithmetic function. Show that f is completely multiplicative if and only if its convolution inverse g is given by the formula $g(n) = \mu(n)f(n)$ for all n.

Problem 2: Given an arithmetic function $f(n)$, we define the Dirichlet series $F(s)$ of $f(n)$ by the formula

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

(The most general way to consider $F(s)$ is as a function of a complex variable, but you can take s to be a real number for the applications we have in mind here.)

(a) Let f and g be two arithmetic functions with associated Dirichlet series F and G. Let s lie in a region in which the series defining $F(s)$ and $G(s)$ converge absolutely. Show that

$$F(s)G(s) = \sum_{n=1}^{\infty} \frac{(f \star g)(n)}{n^s}.$$

(b) Define $\zeta(s)$ to be the Dirichlet series associated to the function $\mathbf{1}$. It is well-known that $\zeta(2) = \pi^2/6$. Deduce that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2}.$$

Problem 3: We have seen that the cyclotomic polynomials $\Phi_n(x)$ are monic irreducible polynomials of degree $\varphi(n)$ with integer coefficients, so they can be written as

$$\Phi_n(x) = x^{\varphi(n)} + c_{\varphi(n)-1}x^{\varphi(n)-1} + \ldots + c_1 x + c_0.$$

For some of these problems, it might help to write down the first ten or twenty cyclotomic polynomials.

(a) Based on your data, make a conjecture about the value of c_0. Prove it.

(b) For $n \geq 2$, show that the coefficients of $\Phi_n(x)$ are palindromic; that is, $c_{\varphi(n)-k} = c_k$.

(c) Based on your data, make a conjecture about the value of c_1. Prove it. (Hint: it might be easier for the proof to analyze $c_{\varphi(n)-1}$ instead, which is equal to c_1 by part (b).)

(d) Show that $\Phi_{na}(x) | \Phi_n(x^a)$ for all a, n.

(e) For which a and n is it true that $\Phi_{na}(x) = \Phi_n(x^a)$? (Hint: Degrees!)

Problem 4: Find all n such that $\varphi(n) = 12$. Prove that your list is complete!

Problem 5: Show that there is a division algorithm in $\mathbb{Z}[x]$ of sorts; more specifically, show that if $a(x)$ and $b(x)$ are polynomials in $\mathbb{Z}[x]$, and $b(x)$ is *monic*, then we can write $a(x) = b(x)q(x) + r(x)$ with $q(x)$ and $r(x)$ in $\mathbb{Z}[x]$, and $\deg r(x) < \deg b(x)$.

Problem 6: Find a formula for the convolution inverse of $\sigma_k(n)$.

Problem 7: This problem is sort of a leftover from a few weeks ago, but its significance will be revisited shortly.

(a) Let p be an odd prime. Show that the following three statements are equivalent:
(1) p can be written in the form $x^2 + 2y^2$, $x, y \in \mathbb{Z}$
(2) -2 is a square mod p
(3) p is not irreducible in the ring $\mathbb{Z}[\sqrt{-2}]$
(Hint: Exercise 9(a) from Homework 1 should help here; also this should remind you of a theorem we proved in class!)

(b) If we really want to make this look like the theorem we proved in class, there should be a fourth equivalent condition, which should be a congruence condition: p is congruent to something(s) mod something. Make a conjecture as to what that condition should be! We will prove it soon. (You'll probably need to decide for which p the condition holds for values of p up to at least 100, to get enough data.)