MATH 4400
HOMEWORK 6
DUE 3/20/08

Instructions: Math 6400 students should do all the problems. Math 4400 students should do seven of the nine problems.

Problem 1: This exercise outlines a proof of Pick's theorem.

(a) Suppose A and B are two lattice polygons which share a side. Let C be the lattice polygon obtained by "gluing" A and B together (so that the side they share vanishes). Show that if Pick's theorem holds for A and B, it holds for C. Also show that if Pick's theorem holds for A and C, it holds for B.

(b) Assuming, then, that every lattice polygon can be partitioned into lattice triangles (which is true and not hard to see), it suffices to prove the theorem for triangles. By drawing a rectangle around an arbitrary triangle, explain why it suffices to prove the theorem for right triangles whose legs are parallel to the coordinate axes, and rectangles whose sides are parallel to the coordinate axes.

(c) Complete the proof of the theorem.

Problem 2: Prove the most general generalization of Minkowski's theorem: let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be linearly independent vectors in $\mathbb{R}^n$, and L be the lattice consisting of all integer linear combinations of the $\mathbf{v}_i$, and Δ be the area of the fundamental parallelopiped of L. Then suppose R is a convex centrally symmetric body of volume $> 2^n\Delta$. Then $R \cap L$ contains at least one point other than the origin.

Problem 3: Let α be an irrational number. We say that p/q is a *good rational approximation* to α if $\left|\alpha - \dfrac{p}{q}\right| < \dfrac{1}{q^2}$. This exercise outlines a proof that every α has infinitely many good rational approximations.

(a) Show that for any real number $m > 1$ which is not an integer, there are integers p, q not both 0 such that

$$|q\alpha - p| < \frac{1}{m}, \quad |q| < m.$$

(Hint: Use Minkowski's theorem! Maybe drawing a picture would help. Make sure you use the assumption that m is not an integer.)

(b) Show that p/q is a good rational approximation to α. Explain why this implies that α has infinitely many good rational approximations.

Problem 4: (taken from an old Putnam exam) Show that any convex lattice pentagon has area $\geq 5/2$. Also, show that this bound is sharp. (Hint for the first part: Pick's theorem!)

Problem 5: Define $s(k)$ to be the smallest positive integer such that k can be written in the form

$$k = x_1^2 \pm \cdots \pm x_{s(k)}^2$$

for some positive integers x_i. (The $\pm$ signs are independent of each other.)

(a) Show that $s(k) \leq 2$ if k is odd and $s(k) \leq 3$ if k is even.

(b) Show that there are infinitely many k such that $s(k) = 3$.

Problem 6: (a) The *Three Squares Theorem* states that a positive integer n can be written as the sum of three squares if and only if $n \neq 4^k(8a + 7)$ for nonnegative integers a, k. Show the "only if" part of this theorem. (The "if" part is much harder.)

(b) Show that the Three Squares Theorem implies Lagrange's Four Squares Theorem (the one we proved in class).

Problem 7: Here we will examine the problem of which integers can be written as sums of five positive squares.

Suppose n cannot be written as a sum of five positive squares. Notice that

$$169 = 13^2 = 12^2 + 5^2 = 12^2 + 4^2 + 3^2 = 11^2 + 4^2 + 4^2 + 4^2 = 12^2 + 4^2 + 2^2 + 2^2 + 1^2.$$

Show that if n cannot be written as the sum of five positive squares, then $n < 169$. (Hint: suppose not; then apply the Four Squares Theorem to $n - 169$.) Computer exercise: check all the numbers less than 169 to find the complete list of positive integers which cannot be written as a sum of five positive squares.

Problem 8: Now we move on to four positive squares. In what follows we will need to assume the Three Squares Theorem (see problem 6).

(a) First show that, for any positive integer k, we have that $8k$ is a sum of four positive squares if and only if $2k$ is.

(b) Show that if $n \equiv 2$, 3, 4, or 6 mod 8, and $n \geq 169$, then n is a sum of four positive squares. (Hint: $n - 169$.)

(c) Suppose that $n \equiv 1$ or 5 mod 8. Show that if $n \geq 676$, then n is a sum of four positive squares. (Hint: $n - 676$.)

(d) Suppose that $n \equiv 7$ mod 8. Show that n is a sum of four positive squares. (Hint: if you apply the correct theorems, this should be no work at all.)

(e) Put it all together to obtain the following result: every positive integer is a sum of four positive squares except for $n = 1, 3, 5, 9, 11, 17, 29, 41$, and all integers of the form $2 \cdot 4^m$, $6 \cdot 4^m$, and $14 \cdot 4^m$ ($m \in \mathbb{Z}$, $m \geq 0$). (Of course this will require a bit of computer searching, much like Problem 7.)

Problem 9: For $k \geq 6$, show that the only positive integers that cannot be written as a sum of k positive squares are

$$1, \ldots, k-1, k+1, k+2, k+3, k+4, k+5, k+7, k+10, k+13.$$