

MATH 4400
HOMEWORK 7
DUE 4/8/08

Instructions: Math 6400 students should do all the problems. Math 4400 students should do six of the seven problems (FYI: the hardest one is probably problem 6).

Problem 1: We define the *mediant* of the fractions $\frac{a}{b}$ and $\frac{c}{d}$ to be $\frac{a+c}{b+d}$.

(a) Let $r_1, s_1, r_2, s_2 \in \mathbb{Q}$. Suppose $r_1 < s_1$ and $r_2 < s_2$. Give an example to show that the mediant of r_1 and r_2 may be greater than the mediant of s_1 and s_2 . Look up *Simpson's paradox* in statistics, and show that this fact about mediants is equivalent to Simpson's paradox.

(b) Show that every term of any Farey sequence is the mediant of its two neighbors.

(c) Now suppose $ad - bc = \pm 1$. Show that the lattice

$$\left\{ r \begin{pmatrix} a \\ b \end{pmatrix} + s \begin{pmatrix} c \\ d \end{pmatrix} : r, s \in \mathbb{Z} \right\}$$

equals \mathbb{Z}^2 .

(d) Show that the lattice points in

$$\left\{ r \begin{pmatrix} a \\ b \end{pmatrix} + s \begin{pmatrix} c \\ d \end{pmatrix} : r, s > 0 \right\}$$

correspond to the fractions between $\frac{a}{b}$ and $\frac{c}{d}$.

(e) Conclude that the fraction between $\frac{a}{b}$ and $\frac{c}{d}$ with the smallest denominator is the mediant of $\frac{a}{b}$ and $\frac{c}{d}$. (Be careful to think about when the fractions in part (d) are in lowest terms.) Deduce the conjecture we mentioned in class, that $\frac{a}{b}$ and $\frac{c}{d}$ are neighbors in the Farey sequence F_n if and only if both of the following conditions hold:

(1) $ad - bc = \pm 1$

(2) $b + d < n$

Problem 2: In this problem we consider the equations $x^2 - 2y^2 = \pm 1$.

(a) Let p_n/q_n be the n th convergent to $\sqrt{2}$. Show that $p_n = 2p_{n-1} + p_{n-2}$ and $q_n = 2q_{n-1} + q_{n-2}$ for $n \geq 1$.

(b) Show that $p_n + q_n\sqrt{2} = (1 + \sqrt{2})^{n+1}$. Deduce that $p_n^2 - 2q_n^2 = (-1)^{n+1}$.

(c) Find closed-form formulas for p_n and q_n . (You will have to solve the recursions in part (a); you might use generating functions, or perhaps some linear algebra, or make a lucky guess and prove it by induction...)

Problem 3: The *triangular numbers* are numbers of the form $n(n+1)/2$ for integer n . The square numbers are numbers of the form m^2 for integer m . Which numbers are both triangular and square? Hint: $2m^2 = n(n+1)$; now complete the square on the right side, and change variables...

Problem 4: Let p_n/q_n be the convergents to the irrational number α . Show that, for all n , the inequality

$$\left| \frac{p}{q} - \alpha \right| < \frac{1}{2q^2}$$

is satisfied by at least one of (p_n, q_n) or (p_{n+1}, q_{n+1}) . (Hint: assume neither one satisfies it and add the inequalities you get.)

Problem 5: Let p_n/q_n be the convergents to the irrational number α . Show that

$$\frac{1}{(a_{n+1} + 2)q_n^2} < \left| \frac{p_n}{q_n} - \alpha \right| < \frac{1}{a_{n+1}q_n^2}.$$

(Hint: all you need is that the convergents alternate sides of α .)

Problem 6: Let d be an integer that is not a perfect square.

(a) Show that the continued fraction for \sqrt{d} is periodic. In fact, show it takes the form $[a_0, \overline{a_1, \dots, a_n}]$.

(b) Show that $a_n = 2a_0$, and that all the other a_k are less than $2a_0$.

(c) Show that the sequence $(p_k^2 - dq_k^2)$ is periodic. What is the period?

(d) Show that the sequence (a_1, \dots, a_{n-1}) is palindromic.

Problem 7: For any positive integer n , find the smallest integer solution to the equation $x^2 - Dy^2 = 1$ for:

(a) $D = n^2 + 1$

(b) $D = n^2 + 2$

(c) $D = n^2 + n$

(d) $D = n^2 - 2$

(e) $D = 9n^2 + 3$

(f) $D = 4n^2 + 7n + 3$