MATH 4400
HOMEWORK 7
COMMENTS AND SOLUTIONS

Problem 1: We define the *mediant* of the fractions $\frac{a}{b}$ and $\frac{c}{d}$ to be $\frac{a+c}{b+d}$.

(a) Let $r_1, s_1, r_2, s_2 \in \mathbb{Q}$. Suppose $r_1 < s_1$ and $r_2 < s_2$. Give an example to show that the mediant of r_1 and r_2 may be greater than the mediant of s_1 and s_2. Look up *Simpson's paradox* in statistics, and show that this fact about mediants is equivalent to Simpson's paradox.

(b) Show that every term in any Farey sequence is the mediant of its two neighbors.

(c) Now suppose $ad - bc = \pm 1$. Show that the lattice

$$\{r \begin{pmatrix} a \\ b \end{pmatrix} + s \begin{pmatrix} c \\ d \end{pmatrix} : r, s \in \mathbb{Z}\}$$

equals $\mathbb{Z}^2$.

(d) Show that the lattice points in

$$\{r \begin{pmatrix} a \\ b \end{pmatrix} + s \begin{pmatrix} c \\ d \end{pmatrix} : r, s > 0\}$$

correspond to the fractions between $\frac{a}{b}$ and $\frac{c}{d}$.

(e) Conclude that the fraction between $\frac{a}{b}$ and $\frac{c}{d}$ with smallest denominator is the mediant of $\frac{a}{b}$ and $\frac{c}{d}$. Deduce the conjecture we mentioned in class, that $\frac{a}{b}$ and $\frac{c}{d}$ are neighbors in the Farey sequence F_n if and only if both of the following conditions hold:

(1) $ad - bc = \pm 1$
(2) $b + d > n$

Solution: (a) Let $r_1 = \frac{12}{48}$, $r_2 = \frac{183}{582}$, $s_1 = \frac{104}{411}$, $s_2 = \frac{45}{140}$. I got these numbers from Wikipedia–they all represent hits divided by at bats in two baseball seasons for two different players. The r_i are Derek Jeter's numbers in 1995 and 1996 respectively, and the s_i are David Justice's. In both seasons, Justice had a higher batting average than Jeter (.253 to .250 in 1995 and .321 to .314 in 1996), but Jeter's combined batting average over the two seasons, which is the mediant of r_1 and r_2, is $\frac{195}{630} = .310$, while Justice's is $\frac{149}{551} = .270$.

This is not a paradox, but it is somewhat counterintuitive, and "Simpson's counterintuitive observation" is not as snappy a name as "Simpson's paradox," so there you go. You'd do well to read about the other real-world examples of this phenomenon that pop up in most introductory statistics books–the Wikipedia entry has a few good ones.

(b) Suppose $\frac{a}{b} < \frac{c}{d} < \frac{e}{f}$ are consecutive terms in a Farey sequence. Then $bc - ad = 1$ and $ed - fc = 1$. Set these equations equal to each other to obtain $c(b + f) = d(a + e)$, so $\frac{c}{d} = \frac{a+e}{b+f}$, which is exactly what we wanted. (Note that this does not mean that $c = a + e$ and $d = b + f$, because $a + e$ and $b + f$ may not be relatively prime.)

(c) Obviously this lattice is contained in $\mathbb{Z}^2$, but it contains

$$d\begin{pmatrix}a\\b\end{pmatrix} - b\begin{pmatrix}c\\d\end{pmatrix} = \begin{pmatrix}\pm 1\\0\end{pmatrix}$$

and

$$c\begin{pmatrix}a\\b\end{pmatrix} - a\begin{pmatrix}c\\d\end{pmatrix} = \begin{pmatrix}0\\\mp 1\end{pmatrix},$$

so it contains all of $\mathbb{Z}^2$.

(d) This is pretty clear when you draw the region that this set covers in $\mathbb{R}^2$. I leave the details to you.

(e) Suppose that we have a fraction $\frac{e}{f}$ between $\frac{a}{b}$ and $\frac{c}{d}$. WLOG suppose $\frac{a}{b} < \frac{c}{d}$, so that $bc - ad = 1$ (we are still under the assumption of part (c)). Then

$$\begin{pmatrix}e\\f\end{pmatrix} = r\begin{pmatrix}a\\b\end{pmatrix} + s\begin{pmatrix}c\\d\end{pmatrix}$$

for some $r, s \in \mathbb{R}$. By part (d), r and s are positive. So we get $ra + sc = e$ and $rb + sd = f$. We can easily solve this to obtain $r = fc - ed$, $s = eb - fa$. The point is that this shows that r and s must then be (positive) integers. Of course this implies that f is minimal when r and s are both equal to 1, as desired. The conjecture follows immediately, as it implies that there is something in F_n between $\frac{a}{b}$ and $\frac{c}{d}$ if and only if $b + d$ is small enough to be a denominator of something in F_n. $\square$

Problem 2: In this problem we consider the equations $x^2 - 2y^2 = \pm 1$.

(a) Let p_n/q_n be the nth convergent to $\sqrt{2}$. Show that $p_n = 2p_{n-1} + p_{n-2}$ and $q_n = 2q_{n-1} + q_{n-2}$ for $n \geq 1$.

(b) Show that $p_n + q_n\sqrt{2} = (1 + \sqrt{2})^{n+1}$. Deduce that $p_n^2 - 2q_n^2 = (-1)^{n+1}$.

(c) Find closed-form formulas for p_n and q_n.

Solution: (a) This is just an immediate consequence of the fact that the continued fraction expansion of $\sqrt{2}$ is $[1, \overline{2}]$.

(b) This we do by induction. The base cases are $n = 0$ and $n = 1$, which are trivial. Now suppose we have proved the result for $n \leq k$; then

$$\begin{aligned}
p_{k+1} + q_{k+1}\sqrt{2} &= (2p_k + p_{k-1}) + (2q_k + q_{k-1})\sqrt{2} \\
&= 2(p_k + q_k\sqrt{2}) + (p_{k-1} + q_{k-1}\sqrt{2}) \\
&= 2(1 + \sqrt{2})^k + (1 + \sqrt{2})^{k-1} \\
&= (1 + \sqrt{2})^{k-1}(2(1 + \sqrt{2}) + 1) \\
&= (1 + \sqrt{2})^{k-1}(3 + 2\sqrt{2}) = (1 + \sqrt{2})^{k+1}.
\end{aligned}$$

The second part of the problem follows by taking the norms of both sides.

(c) The easiest way to do the problem is to apply the automorphism of $\mathbb{Q}(\sqrt{2})$ sending $a + b\sqrt{2}$ to $a - b\sqrt{2}$. As I said, we call this "conjugation," but it's certainly not complex conjugation. Anyway, add (and subtract) the result of (b) to (and from) its conjugate to obtain

$$\begin{aligned} p_n &= \frac{1}{2}((1+\sqrt{2})^{n+1} + (1-\sqrt{2})^{n+1}) \\ q_n &= \frac{1}{2\sqrt{2}}((1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}) \end{aligned}$$

Practically speaking, p_n should be an integer really close to $\frac{1}{2}(1+\sqrt{2})^{n+1}$ and q_n should be an integer really close to $\frac{1}{2\sqrt{2}}(1+\sqrt{2})^{n+1}$. Try it for $n = 10$ to see just how close!

Problem 3: The *triangular numbers* are numbers of the form $n(n+1)/2$ for positive integer n. The square numbers are numbers of the form m^2 for positive integer m. Which numbers are both triangular and square?

Solution: Start with $2m^2 = n(n+1)$. Multiply both sides by 4 and add 1 to obtain $8m^2 + 1 = (2n+1)^2$. So we're looking for solutions to $x^2 - 2y^2 = 1$ where $x = 2n+1$ and $y = 2m$. Notice that any solution to this equation has x odd, and then y is even (mod-8 analysis). So the triangular numbers are precisely $q_{2k+1}^2/4$ for $k \in \mathbb{N}$, where the q_i are the convergents we looked at in the previous problem. The first few are $1, 36, 1225, 41616, \ldots$. Closed-form formulas are not difficult to find. Feel free to google the Encyclopedia of Integer Sequences at this point! $\square$

Problem 4: Let p_n/q_n be convergents to the irrational number α. Show that, for all n, the inequality

$$\left|\frac{p}{q} - \alpha\right| < \frac{1}{2q^2}$$

is satisfied by at least one of (p_n, q_n) or (p_{n+1}, q_{n+1}).

Solution: Suppose it is satisfied by neither. Then

$$\frac{1}{q_n} q_{n+1} = \left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| = \left|\frac{p_n}{q_n} - \alpha\right| + \left|\frac{p_{n+1}}{q_{n+1}} - \alpha\right| \geq \frac{1}{2q_n^2} + \frac{1}{2q_{n+1}^2}$$

and clearing denominators gives $2q_nq_{n+1} \geq q_{n+1}^2 + q_n^2$, or $0 \geq (q_{n+1} - q_n)^2$. But let's go back to the original inequalities; we see that equality can only hold when α is rational. Since this is ruled out by the problem, we can change the $\geq$ to a $>$, and then we get a contradiction. $\square$

Problem 5: Let p_n/q_n be convergents to the irrational number α. Show that

$$\frac{1}{(a_{n+1}+2)q_n^2} < \left|\frac{p_n}{q_n} - \alpha\right| < \frac{1}{a_{n+1}q_n^2}.$$

Solution: The right inequality follows from

$$\left|\frac{p_n}{q_n} - \alpha\right| < \left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| = \frac{1}{q_nq_{n+1}} = \frac{1}{q_n(a_{n+1}q_n + q_{n-1})} \leq \frac{1}{a_{n+1}q_n^2}.$$

The left inequality follows from

$$\begin{aligned}\left|\frac{p_n}{q_n}-\alpha\right| > \left|\frac{p_n}{q_n}-\frac{p_{n+2}}{q_{n+2}}\right| = \frac{a_{n+2}}{q_nq_{n+2}} &= \frac{a_{n+2}}{q_n(a_{n+2}q_{n+1}+q_n)}\\ &= \frac{1}{q_n(q_{n+1}+\frac{1}{a_{n+2}}q_n)}\\ &= \frac{1}{q_n(a_{n+1}q_n+q_{n-1}+\frac{1}{a_{n+2}}q_n)}\\ &\geq \frac{1}{q_n(a_{n+1}q_n+q_n+q_n)} = \frac{1}{(a_{n+1}+2)q_n^2}\end{aligned}$$

Problem 6: Let d be an integer that is not a perfect square.

(a) Show that the continued fraction for $\sqrt{d}$ is periodic. In fact, show that it takes the form $[a_0, \overline{a_1, \ldots, a_n}]$.

(b) Show that $a_n = 2a_0$, and that all the other a_k are less than $2a_0$.

(c) Show that the sequence $(p_k - dq_k^2)$ is periodic. What is the period?

(d) Show that the sequence $(a_1, \ldots, a_{n-1})$ is palindromic.

Solution: (a) We saw in class that a_k is the greatest integer of $\frac{\sqrt{d}+r_k}{s_k}$, where $r_k = (-1)^{k+1}(p_{k-2}p_{k-1}-dq_{k-2}q_{k-1})$ and $s_k = (-1)^k(p_{k-1}^2-dq_{k-1}^2)$. We also saw that $|s_k| < 2\sqrt{d}+1$.

Now $\beta_k = \dfrac{\sqrt{d}+r_k}{s_k}$ is equal to $\dfrac{1}{\beta_{k-1}-a_{k-1}}$, by construction. This "flipping" step of the continued fraction expansion must look like

$$\frac{1}{\beta_{k-1}-a_{k-1}} = \frac{s_{k-1}}{\sqrt{d}-r_k} = \frac{\sqrt{d}+r_k}{s_k} = \beta_k.$$

So we see that $r_k < \sqrt{d}$. And r_k and s_k are always positive (you can probably prove this by induction, if nothing else). Since there are only finitely many possibilities for the ordered pair (r_k, s_k), the sequence of β_k's must be eventually periodic (once it repeats once, it falls into a loop, since β_k determines β_ℓ for all $\ell > k$). Since the sequence of β_k's determines the continued fraction expansion, the continued fraction expansion must be eventually periodic.

We have also seen that, since Pell's equation has solutions, there is some positive integer n such that $s_n = 1$. In that case, $\beta_n = \sqrt{d}+r_n$, and so $\beta_{n+1} = \frac{1}{\sqrt{d}-a_0} = \beta_1$, and so $a_1 = a_{n+1}$ and so on for the rest of the a's.

(c) Actually, we've showed this already, because the s_k are periodic. The point is that the period of the s_k sequence is n, so the period of the sequence we're interested in is n if n is even and $2n$ if n is odd.

(b)(d) Maybe you should ask me about these. Sorry! □

Problem 7: For any positive integer n, find the smallest positive integer solution to the equation $x^2 - Dy^2 = 1$ for $D = n^2+1, n^2+2, n^2+n, n^2-2, 9n^2+3, 4n^2+7n+3$.

Solution: For each of these, we simply find the continued fraction expansion of $\sqrt{D}$. So let us make a list (I leave the computations to you):

$$\begin{aligned}
\sqrt{n^2+1} &= [n, \overline{2n}] \\
\sqrt{n^2+2} &= [n, \overline{n, 2n}] \\
\sqrt{n^2+n} &= [n, \overline{2, 2n}] \\
\sqrt{n^2-2} &= [n-1, \overline{1, n-2, 1, 2n-2}] \\
\sqrt{9n^2+3} &= [3n, \overline{2n, 6n}] \\
\sqrt{4n^2+7n+3} &= [2n+1, \overline{1, 2, 1, 4n+2}]
\end{aligned}$$

Making the magic table in each case, we get:

$$\begin{aligned}
D = n^2+1 &: (2n^2+1)^2 - D(2n)^2 = 1 \\
D = n^2+2 &: (n^2+1)^2 - Dn^2 = 1 \\
D = n^2+n &: (2n+1)^2 - D(2)^2 = 1 \\
D = n^2-2 &: (n^2-1)^2 - Dn^2 = 1 \\
D = 9n^2+3 &: (6n^2+1)^2 - D(2n)^2 = 1 \\
D = 4n^2+7n+3 &: (8n+7)^2 - D(4)^2 = 1
\end{aligned}$$

and that's the smallest one in each case! ☐