Del Pezzo surfaces and the Brauer-Manin obstruction

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Problem: Let X be a (nice) variety over a number field k. Is $X(k)$ empty?

Necessary condition: Consider the set $X(\mathbb{A}_k) = \prod_v X(k_v)$. Then $X(k) \subseteq X(\mathbb{A}_k)$, so if $X(\mathbb{A}_k) = \emptyset$, then $X(k) = \emptyset$.

What about the converse? This is commonly known as the "Hasse Principle." The Hasse Principle holds for quadric hypersurfaces in $\mathbb{P}^n$ (Hasse-Minkowski), but is false for many other common classes of varieties.

Examples:

$3x^3 + 4y^3 + 5z^3 = 0$ in $\mathbb{P}^2$ (Selmer).

$5x^3 + 9y^3 + 10z^3 + 12t^3 = 0$ in $\mathbb{P}^3$ (Cassels-Guy).

It is computationally easy to check whether $X(\mathbb{A}_k)$ is empty, but how does one show that $X(k)$ is empty in general?

Definition: (Manin) Suppose $X(\mathbb{A}_k) \neq \emptyset$. Let $\mathcal{A}$ be an element of Br X. Let

$$X(\mathbb{A}_k)^{\mathcal{A}} = \{(x_v) \in X(\mathbb{A}_k) | \sum_v \mathrm{inv}_v \, \mathcal{A}(x_v) = 0\}.$$

Here $\mathrm{inv}_v \colon \mathrm{Br}\ k_v \to \mathbb{Q}/\mathbb{Z}$ comes from class field theory, and $\mathcal{A}(x_v)$ is the image of $\mathcal{A}$ under the map $\mathrm{Br}\, X \to \mathrm{Br}\, k_v$. Also define

$$X(\mathbb{A}_k)^{\mathrm{Br}} = \bigcap_{\mathcal{A} \in \mathrm{Br}\, X} X(\mathbb{A}_k)^{\mathcal{A}}.$$

From the exact sequence

$$0 \to \mathrm{Br}\ k \to \oplus_v \mathrm{Br}\ k_v \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0,$$

we see that

$$X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}} \subseteq X(\mathbb{A}_k).$$

So, in order to find counterexamples to the Hasse principle, try to find examples of X for which $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$. We say in this case that there is a Brauer-Manin obstruction to rational points on X.

Conjecture: (Colliot-Thélène) The Brauer-Manin obstruction to rational points is the only one for nice rationally connected varieties X.

Proposition 1 *Let*

$\mathrm{Br}_0 X = \mathrm{coker}(\mathrm{Br}\, k \to \mathrm{Br}\, X)$ *("constant algebras")*

$\mathrm{Br}_1 X = \ker(\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X})$ *("algebraic part of the Brauer group").*

Then there is an isomorphism of finite groups

$$\frac{\mathrm{Br}_1 X}{\mathrm{Br}_0 X} \xrightarrow{\sim} H^1(k, \mathrm{Pic}\, \overline{X}).$$

Proof: This comes from a spectral sequence, which is unfortunate for computational purposes because the cokernel is $H^3(k, \overline{k}^*)$; we know this group is zero, but it is not very easy to exhibit a 3-cocycle as a 3-coboundary!

How do we compute the Brauer-Manin obstruction on some class of nice surfaces? Here is a

Program:

1. Find a good set of generators for Pic $\overline{X}$.

2. Use this set to compute the cohomology groups $H^1(k, \mathrm{Pic}\,\overline{X})$.

3. Try to describe algebras in Br X which correspond to nonzero elements of $H^1(k, \mathrm{Pic}\,\overline{X})$.

4. Compute invariants of these algebras.

Even step 1 can be prohibitively difficult, if we do not restrict our attention to sufficiently friendly varieties X.

Examples of the Program in action:

- Diagonal cubic surfaces (Colliot-Thélène, Kanevsky, Sansuc 1980's)

- Del Pezzo surfaces of degree 3 and 4 (Swinnerton-Dyer 1999)

- Diagonal Del Pezzo surfaces of degree 2 (Kresch, Tschinkel 2004)

- More general Del Pezzo surfaces of degree 2 (C. 2005, to appear)

- Diagonal quartic surfaces (Bright 2002)

- Kummer surfaces of a product of two genus-one curves (Argentin 2006)

- Kummer surfaces of 2-coverings of Jacobians of genus-two curves (van Luijk, Logan, in progress)

Definition: A Del Pezzo surface is a nice surface whose anticanonical divisor is ample.

Properties of Del Pezzo surfaces: Each Del Pezzo surface X has a degree d, $1 \leq d \leq 9$.

With one exception, $\overline{X}$ is isomorphic to the blowup of $\mathbb{P}^2$ at $r = 9 - d$ points in general position.

So $\operatorname{Pic} \overline{X} \cong \mathbb{Z}^{r+1}$, generated by the class of a line in the original $\mathbb{P}^2$, together with the exceptional curves over each blown-up point.

When $d = 1, 5, 7$, $X(k)$ is nonempty.

When $d = 6, 8, 9$, X satisfies the Hasse Principle.

When $d = 2, 3, 4$, there are counterexamples to the Hasse Principle (which conjecturally are all explained by the Brauer-Manin obstruction).

Definition: An exceptional curve on $\overline{X}$ is a curve $C \cong \mathbb{P}^1$ such that $(C, C) = -1$.

There are finitely many exceptional curves on $\overline{X}$, and the set $\mathcal{E}$ of such curves is Galois-stable. So we get a map

$$\mathsf{Gal}(\overline{k}/k) \to \mathsf{Aut}(\mathcal{E}) \cong W,$$

where W is a well-known Weyl group, and hence an induced injection $\mathsf{Gal}(M/k) \hookrightarrow W$, where M is the field of definition for the exceptional curves. Using this (and a lot of MAGMA), we can derive the following result:

Theorem 2 *Let X be a Del Pezzo surface of degree d. Then $H^1(k, \mathrm{Pic}\,\overline{X})$ is isomorphic to one of the following groups:*

- 0 $\quad (5 \leq d \leq 9)$
- 0, $\mathbb{Z}/2$, $(\mathbb{Z}/2)^2$ $\quad (d = 4)$
- 0, $\mathbb{Z}/2$, $(\mathbb{Z}/2)^2$, $\mathbb{Z}/3$, $(\mathbb{Z}/3)^2$ $\quad (d = 3)$
- 0, $(\mathbb{Z}/2)^s\,(1 \leq s \leq 6)$, $\mathbb{Z}/3$, $(\mathbb{Z}/3)^2$, $\mathbb{Z}/4 \oplus (\mathbb{Z}/2)^t\,(0 \leq t \leq 2)$, $(\mathbb{Z}/4)^2$ $\quad (d = 2)$
- 0, $(\mathbb{Z}/2)^s\,(1 \leq s \leq 8)$, $(\mathbb{Z}/3)^t\,(1 \leq t \leq 4)$, $\mathbb{Z}/4 \oplus (\mathbb{Z}/2)^u\,(0 \leq u \leq 4)$, $(\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/2)^v$ $(0 \leq v \leq 2)$, $\mathbb{Z}/5$, $(\mathbb{Z}/5)^2$, $\mathbb{Z}/6$, $\mathbb{Z}/6 \oplus \mathbb{Z}/2$, $\mathbb{Z}/6 \oplus \mathbb{Z}/3$, $(\mathbb{Z}/6)^2$ $\quad (d = 1)$

The ideas in the proof of this theorem can be extended to give some information about the rest of the program.

A Del Pezzo surface of degree 2 has the equation $w^2 = F(x, y, z)$, where F is a nonsingular quartic homogeneous polynomial. So there is an associated genus-3 curve $C \subseteq \mathbb{P}^2$ cut out by F.

Proposition 3 *Let X be a Del Pezzo surface of degree 2, so that $W = W(E_7)$. Suppose that $H^1(k, \mathrm{Pic}\,\overline{X})[2] \neq 0$. Then the image of $\mathrm{Gal}(\overline{k}/k)$ in W is contained in a conjugate of one of two subgroups H_{63} and H_{72} of W, where the subscripts denote indices of the subgroups. Moreover, H_{63} is the stabilizer of a nonzero 2-torsion element of Jac C.*

If the nontrivial element of $H^1(k, \mathrm{Pic}\,\overline{X})[2]$ arises from H_{63}, we can describe the corresponding element of $\mathrm{Br}\,X$ quite explicitly in terms of equations for X (using some classical algebraic geometry), and we can use this description to compute the Brauer-Manin obstruction on a wide class of Del Pezzo surfaces with such a 2-torsion element, even when we cannot write down all the exceptional curves explicitly–this is where our method differs from other versions of the general program.

We obtain the following results:

Theorem 4 *Let X be given by the equation*

$$w^2 = Ax^4 + By^4 + Cz^4, \quad A, B, C \in \mathbb{Z} \cap [-50, 50].$$

Then the Brauer-Manin obstruction is the only one for these X.

Theorem 5 *Let X be given by the equation*

$$\begin{aligned} w^2 \; = \; & Ax^4 + By^4 + Cz^4 + 2Dx^2y^2 + 2Ex^2z^2 \\ & +2Gy^2z^2, \; A, B, C, D, E, G \in \mathbb{Z} \cap [-5, 5]. \end{aligned}$$

Then the Brauer-Manin obstruction is the only one for these X.