

AESTHETIC CONSIDERATIONS IN THE  
FOUNDATIONS OF MATHEMATICS

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by

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We know things through their relations. This was clearly pointed out by such philosophers as James, Dewey, and Whitehead. This can be seen in fundamental particle physics, where almost all experiments consist in throwing one particle at another, i.e. scattering theory. Since the latter part of the nineteenth century, set theory has served as a foundation for mathematics. (By foundations of mathematics we mean a single, first order theory from which all familiar mathematical objects can be constructed.) Set theory doesn't take this philosophy to heart. For example, the language of set theory has as its basic predicate elementhood, which suggests to the mind two classes of objects: fundamental objects, or "atoms", and collections of such. For example, one might suppose a basic collection  $U_0$  of indivisible objects, each of which is known into itself and not through its relations to others, from which we would generate all other sets. Another possibility would be an infinite descent, infinite divisibility, i.e. the objects of any set are also sets and the following sequences are possible

$$\dots X_{-n} \in X_{-n+1} \in \dots \in X_u \in \dots \in X_n \in X_{n+1}, \quad X \in X.$$

These two possibilities are usually excluded by axioms because

they don't correspond to our intuitive notion of sets. It is also possible to start with simply the empty set, and to construct all relevant sets from it (construct in an axiomatic sense). This has always seemed a little comical to me, since it suggests that all of mathematics is built on nothing.

From an aesthetic point of view, it would be desirable to have the foundations of mathematics expressed in a language which clearly accepts the concept of knowledge through relations. Professor F. W. Lawvere has not only begun this task, but has already completed most of its essentials. In his papers, "The Category of Categories as a Foundation for Mathematics", "Category-Valued Higher Order Logic", and "Adjointness in Foundations" it can be clearly seen that he was at least partially motivated by these aesthetic considerations.

In a category, an object is known only through its morphisms, i.e. relations. This also holds for the category of categories; hence a category, considered as an object in the category of categories, is known only through its functors. In mathematics, when you restrict yourself to small finite sets, the concept of set and function are intuitive (at least to mathematicians). When you attempt to assail the infinite, it is necessary to follow an axiomatic approach. You can't really "construct" a category, you can't make the category of groups by "putting" in all

groups and group homomorphisms. What must be done is the giving of a set of axioms which one hopes, or sometimes even tries to prove, describes the conceptual idea at hand. In any case, the object under consideration becomes defined implicitly through the axioms.

Category theory contains set theory, i.e. set theory may be reduced to category theory (see Lawvere's paper on the category of sets). Hence, it is clear that the axioms and methods of category theory are at least as strong, and I feel much stronger, than those of set theory. The modern theory of differential geometry can be obtained from category theory by factoring through set theory. But this is aesthetically unpleasing. In Euclidian geometry, as reformulated by Hilbert, point, line, and plane are undefined entities; a line is not considered as a set of points, but points are said to lie on a line. In modern differential geometry, a differentiable manifold must be defined as a set of points with certain structure. This puts the point in a very special position. It would be much more pleasing to be able to describe the category of differentiable manifolds and differentiable maps intrinsically. From a mathematical point of view this would probably add a deeper insight into the nature of manifolds. This type of program has already been carried out for sets, categories, and topological spaces.

My own specialty is mathematical physics, which I

feel will soon be receiving a large dose of category theory.

My last comment is a sociological one. The particular language used has an important effect on the type of questions and problems that are considered. It affects the whole aesthetic of the subject and shouldn't be underestimated. It is presently possible to reach the frontiers of research in category theory by reading the single survey of Saunders MacLane entitled "Categorical Algebra".

DAE:tb  
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