

NEF DIVISORS ON $\overline{M}_{0,n}$ FROM GIT

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ABSTRACT. We introduce and study *the GIT cone* of $\overline{M}_{0,n}$, which is generated by the pullbacks of the natural ample line bundles on the GIT quotients $(\mathbb{P}^1)^n // \mathrm{SL}(2)$. As one application, we prove unconditionally that the log canonical models of $\overline{M}_{0,n}$ with a symmetric boundary divisor coincide with the moduli spaces of weighted curves or with the symmetric GIT quotient, extending the result of Matt Simpson.

1. INTRODUCTION

The moduli space of smooth pointed genus zero curves $M_{0,n}$ has many compactifications. Among the most important of these is $\overline{M}_{0,n}$, the moduli space of Deligne-Mumford stable curves. There are many beautiful results on these spaces; for instance, Keel and Kapranov described them as explicit blowups of $(\mathbb{P}^1)^{n-3}$ and \mathbb{P}^{n-3} ([Kee92, Kap93b]), and Kapranov identified $\overline{M}_{0,n}$ with the Chow quotient of the Grassmannian $\mathbf{Gr}(2, n)$ by a torus. But the birational geometry of these spaces is still not fully understood, and it is known to be very complicated. For example, Faber computed that the nef cone of $\overline{M}_{0,6}$ has 3190 extremal rays.

Another family of compactifications of $M_{0,n}$ is provided by the GIT quotients $(\mathbb{P}^1)^n //_{\vec{x}} \mathrm{SL}(2)$, where the vector \vec{x} specifies the linearization $\mathcal{O}(x_1, \dots, x_n)$. Let $\mathcal{L}_{\vec{x}}$ denote the distinguished polarization on the GIT quotient descending from the linearization.

By [Kap93a], there are birational morphisms $\pi_{\vec{x}} : \overline{M}_{0,n} \rightarrow (\mathbb{P}^1)^n //_{\vec{x}} \mathrm{SL}(2)$, and hence the pullbacks $L_x := \pi_{\vec{x}}^* \mathcal{L}_{\vec{x}}$ of the distinguished polarizations on the GIT quotients are big and nef, since $\pi_{\vec{x}}$ is birational and $\mathcal{L}_{\vec{x}}$ is ample. In this paper we consider the subcone generated by the $L_{\vec{x}}$ inside the nef cone of $\overline{M}_{0,n}$. We call this *the GIT cone* and denote it **GC**.

Our first main result (see Section 2) tells how the line bundles $L_{\vec{x}}$ intersect the vital curves in $\overline{M}_{0,n}$. This allows us to describe a finite set of generators for the GIT cone, which we call the *GIT complex 0-cells*. We show that for $n \geq 6$ **GC** is strictly smaller than the nef cone.

The symmetric group S_n acts on $\overline{M}_{0,n}$ by permuting the points, and symmetric divisors often play a key role. Let $S_{\vec{x}}$ denote the symmetrization of $L_{\vec{x}}$, that is, $S_{\vec{x}} := \bigotimes_{\sigma \in S_n} L_{\sigma \vec{x}}$. We call the cone generated by these the *symmetrized GIT cone*, and denote it **SGC**. We give a conjectural list of extremal rays of **SGC** and give an example showing that **SGC** is strictly smaller than the symmetric nef cone.

Thus, the $L_{\vec{x}}$ and $S_{\vec{x}}$ do not completely describe the Mori theory of $\overline{M}_{0,n}$. Nevertheless, we believe these cones are very useful for the study of $\overline{M}_{0,n}$. As an illustration, we reprove a result of Simpson and Fedorchuk–Smyth.

Matt Simpson proved the following results in his dissertation: Assume the S_n -equivariant Fulton Conjecture (which is known for $n \leq 24$ [Gib03]). If certain divisors A_{α} (defined below) on $\overline{M}_{0,n}$ are nef, then the log canonical models of the pair $(\overline{M}_{0,n}, \Delta)$ can be identified either as $\overline{M}_{0,\beta}$ for symmetric sets of weights β or as $(\mathbb{P}^1)^n // \mathrm{SL}(2)$ with the symmetric linearization (cf. [Sim08] Corollary 2.3.5, Theorem 2.4.5.) Simpson then proves that these divisors A_{α} are F-nef ([Sim08] Proposition 2.4.6). Using the S_n -equivariant Fulton conjecture again, this yields as a corollary that the log canonical models are indeed the spaces claimed.

In Section 3, we express Simpson's divisors A_α as explicit effective combinations of a small number of the $L_{\vec{x}}$. In particular, this proves that Simpson's A_α are nef without using the S_n -equivariant Fulton Conjecture. This removes one of his two uses of the S_n -equivariant Fulton Conjecture. In Section 4, we prove that A_α is ample on the appropriate $\overline{M}_{0,\beta}$, removing Simpson's second use of the S_n -equivariant Fulton Conjecture.

Remark. Fedorchuk and Smyth have recently given a proof ([FS08]) of Simpson's result which is also fully independent of the S_n -equivariant Fulton Conjecture.

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2. THE MAIN LINE BUNDLES AND THEIR INTERSECTION THEORY

Let \vec{x} be a vector in \mathbb{Q}^n such that $0 < x_i \leq 1$ for all i and $\sum_{i=1}^n x_i = 2$. Then \vec{x} specifies a linearization on $(\mathbb{P}^1)^n$, and hence a distinguished ample line bundle $\mathcal{L}_{\vec{x}}$ on the GIT quotient $(\mathbb{P}^1)^n //_{\vec{x}} \mathrm{SL}(2)$. On the other hand, \vec{x} gives a linearization for the action of $T = \mathbb{G}_m^n / \mathrm{diag} \mathbb{G}_m$ on $\mathbf{Gr}(2, n)$, and by [Kap93a] one has the Gelfand-MacPherson correspondence $(\mathbb{P}^1)^n //_{\vec{x}} \mathrm{SL}(2) \cong \mathbf{Gr}(2, n) //_{\vec{x}} T$ (see [HMSV05] for a nice exposition).

There is a birational morphism $\pi_{\vec{x}} : \overline{M}_{0,n} \rightarrow (\mathbb{P}^1)^n //_{\vec{x}} \mathrm{SL}(2)$ ([Kap93b], [Kap93a], [Has03]).

Definition 2.1. $L_{\vec{x}} := \pi_{\vec{x}}^* \mathcal{L}_{\vec{x}}$.

The line bundle $L_{\vec{x}}$ is nef, since it is the pullback of an ample bundle under a morphism. If all $x_i < 1$ then the morphism $\pi_{\vec{x}}$ is birational, and then $L_{\vec{x}}$ is also big.

We can extend the definition to cover the case when some coordinates of \vec{x} are zero. Start with $n = 4$. Then if any $x_i = 0$, $L_{\vec{x}}$ is the trivial line bundle. If $n \geq 4$ and $x_i = 0$, then define $L_{\vec{x}} = \pi_i^* L_{\hat{\vec{x}}}$, where $\pi_i : \overline{M}_{0,n} \rightarrow \overline{M}_{0,n-1}$ is the morphism which forgets the i^{th} marked point and $\hat{\vec{x}}$ is the vector obtained from \vec{x} by omitting the i^{th} coordinate. We also formally set $L_{\vec{x}} = 0$ if any $x_i = 1$.

The following lemma allows us to compute intersections of the $L_{\vec{x}}$ with vital curves:

Lemma 2.2. *Let $\{a, b, c, d\}$ be a partition of the set $\{1, \dots, n\}$ into four nonempty subsets, and let $[C(a, b, c, d)]$ be the corresponding vital curve class. Let \vec{x} be a set of weights as above. Write $x_a = \sum_{i \in a} x_i$, $x_b = \sum_{i \in b} x_i$, etc. We abbreviate*

$$\min := \min\{x_a, x_b, x_c, x_d\}, \quad \max := \max\{x_a, x_b, x_c, x_d\}.$$

Then

$$(1) \quad L_{\vec{x}} \cdot [C(a, b, c, d)] = \begin{cases} 0 & \text{if } \max \geq 1, \\ 2(1 - \max) & \text{if } \max \leq 1 \text{ and } \max + \min \geq 1, \\ 2 \min & \text{if } \max \leq 1 \text{ and } \max + \min \leq 1. \end{cases}$$

Proof. We reduce the calculation to $\overline{M}_{0,4}$ and exploit the fact that $\overline{M}_{0,4} \cong \mathbb{P}^1$ is in a natural way a conic in the toric variety \mathbb{P}^2 . For simplicity of notation, we assume that the partition $\{a, b, c, d\}$ is ordered, i.e. $a = \{1, \dots, \#a\}$, $b = \{\#a + 1, \dots, \#a + \#b\}$, etc.

Recall ([Kee92],[KM96]) that a vital curve in $\overline{M}_{0,n}$ parametrizes nodal curves whose dual graph is a trivalent tree except at one vertex, which is 4-valent. As the cross ratio of the four nodes on the corresponding component varies, a \mathbb{P}^1 in $\overline{M}_{0,n}$ is swept out. The 4-valent vertex partitions in the n leaves into 4 groups, and this is recorded by a partition $\{a, b, c, d\}$. Two vital curves are linearly equivalent if and only if they have the same partition, so the partition specifies a class $[C(a, b, c, d)]$.

Now consider the map $(\mathbb{P}^1)^4 \rightarrow (\mathbb{P}^1)^n$ which is the product of the diagonal map $\Delta_a : \mathbb{P}^1 \rightarrow (\mathbb{P}^1)^a$ and the similar maps $\Delta_b, \Delta_c, \Delta_d$. The pullback of the linearization \vec{x} on $(\mathbb{P}^1)^n$ to $(\mathbb{P}^1)^4$ is $\vec{x}' = (x_a, x_b, x_c, x_d)$. Therefore, the pullback of $\mathcal{L}_{\vec{x}}$ to $(\mathbb{P}^1)^4 //_{\vec{x}'} \mathrm{SL}(2)$ is $\mathcal{L}_{\vec{x}'}$. This gives the following

commutative diagram:

$$\begin{array}{ccc} \overline{M}_{0,n} & \xrightarrow{\pi_{\vec{x}}} & (\mathbb{P}^1)^n //_{\vec{x}} \mathrm{SL}(2) \\ \uparrow i & & \uparrow i_{a,b,c,d} \\ \overline{M}_{0,4} & \xrightarrow{\cong} & (\mathbb{P}^1)^4 //_{\vec{x}'} \mathrm{SL}(2) \end{array}$$

in which the lower arrow is an isomorphism because both varieties are isomorphic to \mathbb{P}^1 . The class of $i_*[\overline{M}_{0,4}]$ in $\overline{M}_{0,n}$ is $[C(a, b, c, d)]$. Thus, to complete the proof we just need to compute the degree of the sheaf $\mathcal{L}_{\vec{x}'}$ on \mathbb{P}^1 . By the Gelfand-MacPherson correspondence

$$(\mathbb{P}^1)^4 //_{\vec{x}'} \mathrm{SL}(2) \cong \mathbf{Gr}(2, 4) //_{\vec{x}'} T$$

where $T = \mathbb{G}_m^4 / \mathrm{diag} \mathbb{G}_m$. The Plücker embedding descends to a map of torus quotients

$$\mathbf{Gr}(2, 4) //_{\vec{x}'} T \hookrightarrow \mathbb{P}^5 //_{\vec{x}'} T$$

The GIT quotient $\mathbb{P}^5 //_{\vec{x}'} T$, together with the polarization given by the GIT construction, is the polarized toric variety corresponding to the fiber over the point $\vec{x}' = (x_a, x_b, x_c, x_d)$ of the polytopal map from the simplex $\sigma_6 \subset \mathbb{R}^6$ with 6 vertices to the hypersimplex $\Delta(2, 4)$ to \mathbb{R}^4 .

Denote the coordinates in \mathbb{R}^6 by $z_{ij} = z_{ji}$ with $i \neq j$, and the coordinates in \mathbb{R}^4 by x_i . Then the map is given by $x_i = \sum_{j \neq i} z_{ij}$. Assume that $x_a \geq x_b \geq x_c \geq x_d$ and that $x_a + x_b \geq x_c + x_d$. Then an easy explicit computation shows that if $x_a > 1$ then the fiber over \vec{x}' is empty, and if $x_a \leq 1$ then it is the triangle with the vertices

$$\begin{aligned} & (x_2, (x_1 - x_2 + x_3 - x_4)/2, (x_1 - x_2 - x_3 + x_4)/2, 0, 0, (-x_1 + x_2 + x_3 + x_4)/2) \\ & ((x_1 + x_2 - x_3 - x_4)/2, x_3, (x_1 - x_2 - x_3 + x_4)/2, 0, (-x_1 + x_2 + x_3 + x_4)/2, 0) \\ & ((x_1 + x_2 - x_3 - x_4)/2, (x_1 - x_2 + x_3 - x_4)/2, x_4, (-x_1 + x_2 + x_3 + x_4)/2, 0, 0) \end{aligned}$$

This polytope is a standard triangle, shifted and dilated by a factor $\lambda = 1 - x_a = 1 - \max$. Thus, $(\mathbb{P}^5, \mathcal{O}(1)) //_{\vec{x}'} T = (\mathbb{P}^2, \mathcal{O}(\lambda))$. The quotient $\mathbf{Gr}(2, 4) // T$ is a conic in this \mathbb{P}^2 and $\mathcal{L}_{\vec{x}'} = \mathcal{O}_{\mathbb{P}^1}(2\lambda)$.

This proves the formula in the first two cases. The third case holds by symmetry. \square

Remark. Let (x_a, x_b, x_c, x_d) be a point in $\Delta(2, 4)$. Define $d(x_a, x_b, x_c, x_d)$ to be the distance from (x_a, x_b, x_c, x_d) to the boundary of $\Delta(2, 4)$. Then $L_{\vec{x}} \cdot [C(a, b, c, d)]$ is a multiple of $d(x_a, x_b, x_c, x_d)$.

2.1. Notation for symmetrized line bundles. It is natural to work with symmetrizations of the $L_{\vec{x}}$. We use the following notation for these:

Definition 2.3. We denote the symmetrization of any $L_{\vec{x}}$ by $S_{\vec{x}}$:

$$S_{\vec{x}} := \bigotimes_{\sigma \in S_n} L_{\sigma \vec{x}}.$$

For some special sets of weights, we have additional notation. Let a be a rational number such that $\frac{1}{n-1} \leq a \leq \frac{2}{n-1}$, and write $\bar{a} = 2 - (n-1)a$. This condition on a ensures that $0 \leq a \leq \bar{a} \leq 1$. We write $L(a, i)$ for $L_{\vec{x}}$ where $x_j = a$ if $j \neq i$ and $x_i = \bar{a}$, and call the i^{th} entry the odd entry. We write

$$V(a, n) := \bigotimes_{i=1}^n L(a, i)$$

Thus, $V(a, n)$ is a reduced symmetrization:

$$V(a, n) = \frac{1}{(n-1)!} S_{\vec{x}}.$$

Note that when $a = 2/n$, we have $a = \bar{a}$; our convention in this case is that $V(a, n) = L_{(a, \dots, a)}^{\otimes n}$ (that is, we do not further reduce the symmetrization).

2.2. Results on the GIT cone and symmetrized GIT cone. Suppose $\vec{x} \in \Delta(2, n)$ with $x_i \neq 0$ for any i . The GIT stability criterion for points on a line is well-known: $(P_1, \dots, P_n) \in (\mathbb{P}^1)^n$ is $\mathrm{SL}(2)$ -stable (resp. semistable) with respect to the linearization \vec{x} if and only if for every subset $I \subseteq \{1, \dots, n\}$ for which the points $\{P_i \mid i \in I\}$ coincide, $\sum_{i \in I} x_i < 1$ (resp. ≤ 1). Thus, for a generic linearization, stability and semistability coincide; strictly semistable points only occur for those \vec{x} which lie on a hyperplane of the form $\sum_{i \in I} x_i = 1$ for some subset $I \subseteq \{1, \dots, n\}$. This is the familiar wall-and-chamber decomposition of the space of linearizations, due to Brion-Procesi, Dolgachev–Hu and Thaddeus. We call the subdivision of $\Delta(2, n)$ given by these hyperplanes *the GIT complex*.

The GIT cone is generated by the 0-cells of the GIT complex because $L_{\vec{x}} \cdot [C(a, b, c, d)]$ is linear on each chamber.

For $n = 5$, we can easily check that the GIT cone is the nef cone. The 0-cells for $n = 5$ are of the form $(2/3, 1/3, 1/3, 1/3)$, $(1/2, 1/2, 1/2, 1/2, 0)$, $(1, 1, 0, 0, 0)$, or the permutations of these. The line bundles of type $(1, 1, 0, 0, 0)$ are trivial. The line bundles of type $(2/3, 1/3, 1/3, 1/3)$ each contract four disjoint curves, giving \mathbb{P}^2 . The line bundles of type $(1/2, 1/2, 1/2, 1/2, 0)$ give morphisms to \mathbb{P}^1 .

But for $n \geq 6$ we see that the GIT cone is strictly contained in the nef cone. Indeed, when $n = 6$, Faber computes that the nef cone has 3190 extremal rays ([Fab00]). We computed the 0-cells of the GIT complex by a search implemented in M2. The interior 0-cells are $(1/3, 1/3, 1/3, 1/3, 1/3, 1/3)$, $(1/4, 1/4, 1/4, 1/4, 1/4, 3/4)$, $(1/4, 1/4, 1/4, 1/4, 1/2, 1/2)$, $(3/5, 2/5, 2/5, 1/5, 1/5, 1/5)$ and their symmetric images, and the boundary 0-cells are $(1, 1, 0, 0, 0, 0)$, $(1/2, 1/2, 1/2, 1/2, 0, 0)$, and $(2/3, 1/3, 1/3, 1/3, 1/3, 0)$ and their symmetric images. There are only 192 in total, so there are not nearly enough to generate the nef cone.

We have a more precise conjectural description of the symmetrized GIT cone:

Conjecture 2.4. $\mathrm{SGC} = \langle V(a, n) \mid a = \frac{1}{t}, t \in \mathbb{Z}, \lfloor \frac{n}{2} \rfloor \leq t \leq n - 2 \rangle$

For $n = 6$, we can check the conjecture. For symmetric divisors, the shape of the partition $\{a, b, c, d\}$ determines the intersection $D \cdot [C(a, b, c, d)]$, so we need only record the intersections with $C(3, 1, 1, 1)$ and $C(2, 2, 1, 1)$. These are as follows:

	$V(1/3, 6)$	$V(1/4, 6)$
$C(2, 2, 1, 1)$	4	1
$C(3, 1, 1, 1)$	0	$3/2$

Then if $(S_{\vec{x}} \cdot C(2, 2, 1, 1), S_{\vec{x}} \cdot C(3, 1, 1, 1)) = (a, b)$, we can write $S_{\vec{x}} \equiv c_1 V(1/3, 6) + c_2 V(1/4, 6)$, where

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 0 & 3/2 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{4}a - \frac{1}{6}b \\ \frac{2}{3}b \end{pmatrix}.$$

$\frac{2}{3}b$ is automatically nonnegative, since $S_{\vec{x}}$ is nef. So the conjecture will be true if $\frac{1}{4}a - \frac{1}{6}b \geq 0$ for any \vec{x} . But

$$F(\vec{x}) = \frac{1}{4}a - \frac{1}{6}b = \frac{1}{4}(S_{\vec{x}} \cdot C(2, 2, 1, 1)) - \frac{1}{6}(S_{\vec{x}} \cdot C(3, 1, 1, 1))$$

is a piecewise linear function which breaks only along the hyperplanes $\sum_{i \in I} x_i = 1$. Hence F attains its minimum on a 0-cell of the GIT complex. We check all the 0-cells for $n = 6$ and see that $F \geq 0$ for all of them.

We give an example showing that **SGC** is strictly contained in the symmetric nef cone for $n = 6$. The F-conjecture is known for $n = 6$ ([Fab00], [KM96], [GKM02]), so any line bundle which intersects the vital curves nonnegatively is nef. Suppose $D := r_2 D_2 + r_3 D_3$ is a symmetric divisor on $\overline{M}_{0,6}$. Then by [KM96] Corollary 4.4, $(D \cdot C(2, 2, 1, 1), D \cdot C(3, 1, 1, 1)) = (2r_3 - r_2, 3r_2 - r_3)$, and D is nef provided $2r_3 - r_2 \geq 0$ and $3r_2 - r_3 \geq 0$, or $2r_3 \geq r_2 \geq (1/3)r_3$. But there are values of

(r_2, r_3) for which $2r_3 \geq r_2 \geq (1/3)r_3$ but $\frac{1}{4}a - \frac{1}{6}b = 8r_3 - 9r_2 < 0$. (For instance, $(r_2, r_3) = (1, 1)$.) So for $n = 6$ the **SGC** is strictly contained in the symmetric nef cone.

3. SIMPSON'S DIVISORS ARE EFFECTIVE COMBINATIONS OF THE $V(a, n)$

3.1. The divisors A_α . First, we recall the key divisors A_α from Simpson's work. Write $D_j = \sum_{|S|=j} D_S$. Then one basis for the S_n -equivariant divisors on $\overline{M}_{0,n}$ is $\{D_2, \dots, D_{\lfloor \frac{n}{2} \rfloor}\}$. Simpson establishes a formula for the A_α in this basis, which we will use as the definition.

Definition 3.1 (cf. [Sim08] Def. 2.3.2 and Lemma 2.3.1). *Write*

$$(2) \quad k(\alpha) := \left\lfloor \frac{2}{\alpha} \right\rfloor - 1.$$

Then

$$(3) \quad A_\alpha = \sum_{j=2}^k \left(\binom{j}{2} \alpha - \frac{j(j-1)}{n-1} \right) D_j + \sum_{j=k+1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{(j-2)(n-1) - j(j-1)}{n-1} + \alpha \right) D_j.$$

Simpson observes that if $\alpha \in [\frac{2}{k+2}, \frac{2}{k+1}]$, then A_α is a convex combination of $A_{2/(k+2)}$ and $A_{2/(k+1)}$, and hence:

Claim 3.2. *It is sufficient to show that A_α is nef for $\alpha = \frac{2}{k+1}$ where $k = 2, \dots, \lfloor \frac{n}{2} \rfloor - 1$. We call these the critical α and A_α .*

Remark. It is easy to check that $A_{2/3} = \frac{1}{3}A_1 + \frac{2}{3}A_{1/2}$. However, this will not affect any of the analysis below, and it does not appear that any of the other critical A_α is a convex or effective combination of the others.

3.2. When n is odd. It is convenient to use the following notation.

Definition 3.3. *Write*

$$(4) \quad f := \left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$$

$$(5) \quad \ell := f + 1 - k.$$

We chose the letter f for floor.

Proposition 3.4. *Suppose n is odd, and that $\alpha = \frac{2}{k+1}$ for some k in $\{1, \dots, \lfloor (n-1)/2 \rfloor\}$. Hence $\alpha = \frac{2}{f-\ell+2}$, and $\ell \in \{1, \dots, f-1\}$. Then*

$$(6) \quad A_\alpha \equiv c_1 V\left(\frac{1}{f+1}, n\right) + \dots + c_{f-1} V\left(\frac{1}{2f-1}, n\right)$$

where

$$(7) \quad c_i = \alpha \frac{(f+i)(f-\ell+1)}{(f-i)(f-i+1)(f-i+2)}$$

if $1 \leq i \leq \ell - 1$,

$$(8) \quad c_\ell = \frac{1}{4} \alpha \frac{(f+\ell)(f-\ell)}{f-\ell+2},$$

and $c_i = 0$ for $\ell + 1 \leq i \leq f - 1$. This covers all critical α except $\alpha = 1$.

For $\alpha = 1$, we have

$$(9) \quad c_i = \frac{f+i}{(f-i)(f-i+1)(f-i+2)}$$

for $1 \leq i \leq f - 1$.

Remark. Since α is of order $1/n$, we see that for $\alpha < 1$ all but one of the coefficients are of order $1/n^2$, while the last coefficient is of order 1. When $\alpha = 1$, all of the coefficients are of order $1/n^2$.

Remark. By Simpson's work, we know $\overline{M}_{0,(1/k,\dots,1/k)}$ corresponds to A_α where $\alpha = 2/(k+1)$. Then the nonzero coefficients in the expression above are $V(1/f, n), V(1/(f+1), n), \dots, V(1/(n-k), n)$. It is convenient to order these 1 through ℓ where $\ell = f+1-k$.

Proof. We choose a set of vital curves C_1, \dots, C_{f-1} (see Definition 3.5 below) which give a full rank intersection matrix $(C_i \cdot V(\frac{1}{f+j}, n))$ for symmetric divisors on $\overline{M}_{0,n}$. Then the c_i are the solutions of the system

$$(10) \quad \left(C_i \cdot V\left(\frac{1}{f+j}, n\right) \right) \vec{c} = (C_i \cdot A_\alpha).$$

In Lemmas 3.6 and 3.7 below, we show that the system (10) takes on the following form:

$$(11) \quad \begin{pmatrix} \frac{2(f+1)}{f+1} & \frac{4}{f+2} & \frac{4}{f+3} & \cdots & \frac{4}{f+j} & \cdots & \frac{4}{2f-1} \\ 0 & \frac{2f}{f+2} & \frac{4}{f+3} & \vdots & \vdots & \vdots & \frac{4}{2f-1} \\ 0 & 0 & \frac{2(f-1)}{f+3} & \vdots & \vdots & \vdots & \frac{4}{2f-1} \\ \vdots & \vdots & 0 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \frac{2(f-j+2)}{f+j} & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \frac{4}{2f-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{2f-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_{f-1} \end{pmatrix} = \begin{pmatrix} \alpha \\ \vdots \\ \alpha \\ 1-\alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \leftarrow (\ell-1)^{th} \text{ coordinate} \\ \leftarrow \ell^{th} \text{ coordinate} \end{matrix}$$

We will refer to the upper triangular matrix on the left hand side of (11) as U and write $u_{i,j}$ for its entries. The vector shown on the right hand side of (11) is for $\alpha < 2/3$. For $\alpha = 2/3$, the vector $(C_i \cdot A_\alpha)$ on the right hand side has no trailing zeroes, and for $\alpha = 1$, the right hand side vector is $(1, \dots, 1)^T$.

Thus, to prove Proposition 3.4, it remains only to show that the c_i given in the statement of the proposition are indeed the solutions of the system (11). This is done in Lemmas 3.8 and 3.9. \square

Definition 3.5. *The set of vital curves we use to define the $(f-1) \times (f-1)$ system above is as follows:*

$$(12) \quad \begin{aligned} C_1 &:= C(f, f-1, 1, 1) \\ C_2 &:= C(f+1, f-2, 1, 1) \\ &\vdots \\ C_i &:= C(f+i-1, f-i, 1, 1) \\ &\vdots \\ C_{f-1} &:= C(f+1+(f-1), f-(f-1), 1, 1) = C(n-3, 1, 1, 1) \end{aligned}$$

where for each i we may choose any partition of $\{1, \dots, n\}$ which has the indicated shape.

The next lemma establishes that the intersection matrix $C_i \cdot V(\frac{1}{f+j}, n)$ is the matrix U shown in (11).

Lemma 3.6. For any $i, j \in \{1, \dots, f-1\}$,

$$C_i \cdot V\left(\frac{1}{f+j}, n\right) = u_{i,j} = \begin{cases} \frac{4}{f+j} & \text{if } i < j \\ \frac{2(f-j+2)}{f+j} & \text{if } i = j \\ 0 & \text{if } i > j \end{cases}$$

Proof. Note that the odd entry of $V(1/(f+j), n)$ is $2 - (n-1)/(f+j) = 2j/(f+j)$. We compute the intersection of C_i with each $L_{\overline{x}}$ constituting $V(1/(f+j), n)$ and sum these to obtain $C_i \cdot V(1/(f+j), n)$.

Case 1: $i < j$. First suppose the odd entry of $L_{\overline{x}}$ lands in the long tail of C_i . Then, using the notation of Lemma 2.2,

$$x_a = (f+i)\frac{1}{f+j} + \frac{2j}{f+j} > 1,$$

so $C_i \cdot L_{\overline{x}} = 0$.

Next suppose the odd entry of $L_{\overline{x}}$ is on the second tail of C_i . Then

$$x_b = (f-i-1)\frac{1}{f+j} + \frac{2j}{f+j} \geq 1,$$

so $C_i \cdot L_{\overline{x}} = 0$.

Thus we see that $L_{\overline{x}}$ contributes to $C_i \cdot V(1/(f+j), n)$ only if the odd entry is on the spine of C_i . The maximum comes from either the long tail, which has weight $x_a = (f+i-1)/(f+j)$, or the odd entry, which has weight $x_c = 2j/(f+j)$. It seems that both are possible. The minimum is $x_d = 1/(f+j)$. However, since both $x_a < 1$ and $x_c < 1$, and both $x_a + \min \leq 1$ and $x_b + \min \leq 1$, we find that $C_i \cdot L_{\overline{x}} = 2 \min = 2/(f+j)$.

Thus, $C_i \cdot V(1/(f+j), n) = 2 \cdot 2/(f+j) = 4/(f+j)$.

Case 2: $i = j$. If the odd entry is on the long tail,

$$x_a = (f-i-2)\frac{1}{f+j} + \frac{2j}{f+j} = \frac{f+3j-2}{f+j} > 1,$$

so $C_i \cdot L_{\overline{x}} = 0$.

If the odd entry is on the second tail,

$$x_b = (f-j-1)\frac{1}{f+j} + \frac{2j}{f+j} = \frac{f+j-1}{f+j} < 1.$$

Meanwhile, on the long tail,

$$x_a = \frac{f+j-1}{f+j},$$

so in the notation of Lemma 2.2 $\max = x_a = x_b$, and $\max + \min = 1$, so $C_i \cdot L_{\overline{x}} = 2 \min = 2/(f+j)$.

Finally, if the odd entry is on the spine, then the long tail gives the maximum, and $\max + \min = 1$, so $C_i \cdot L_{\overline{x}} = 2 \min = 2/(f+j)$.

Thus,

$$C_i \cdot V(1/(f+j), n) = (f-i)\frac{2}{f+j} + 2\frac{2}{f+j} = \frac{2(f-j+2)}{f+j}$$

Case 3: $i > j$. If the odd entry is not on the long tail, then

$$x_a = (f+i-1)\frac{1}{f+j} \geq 1$$

and $C_i \cdot L_{\overline{x}} = 0$.

If the odd entry is on the long tail, then x_a is even bigger, and $C_i \cdot L_{\overline{x}} = 0$. Thus, $C_i \cdot V(1/(f+j), n) = 0$. \square

In the next lemma we compute the right hand side of (11).

Lemma 3.7. *Let $\alpha = \frac{2}{f-\ell+2}$ for $\ell \in \{1, \dots, f-1\}$. Then the intersection numbers $C_i \cdot A_\alpha$ are as follows:*

$$(13) \quad C_i \cdot A_\alpha = \begin{cases} \alpha & \text{if } i < \ell, \\ 1 - \alpha & \text{if } i = \ell, \\ 0 & \text{if } i > \ell. \end{cases}$$

For $\alpha = 1$, we have $C_i \cdot A_1 = 1$ for all i .

Proof. It is lengthy but straightforward to compute this using Definition 3.1 and [KM96] Corollary 4.4. \square

We have now identified the systems (10) and (11).

Next we prove that the c_i given in the statement of Proposition 3.4 are indeed the solutions of this system. We will use the following identity:

Lemma 3.8. *The following identity holds for rational functions of a single variable y :*

$$(14) \quad \sum_{p=1}^m \frac{1}{(y-p)(y-p+1)(y-p+2)} = \frac{2ym - m^2 + m}{2y(y+1)(y-m)(y-m+1)}.$$

Proof. By induction on m . The induction step is easily verified by hand or with a computer algebra system. \square

Lemma 3.9. *The c_i defined in (7) and (8) for $\alpha < 1$, or in (9) for $\alpha = 1$, are the solutions of the system (11).*

Proof. We give the proof for $\alpha < 1$. The proof for $\alpha = 1$ is similar.

It is easy to verify that

$$(15) \quad \begin{aligned} u_{\ell, \ell} c_\ell &= \frac{f - \ell}{f - \ell + 2} \\ u_{\ell-1, \ell-1} c_{\ell-1} + u_{\ell-1, \ell} c_\ell &= \alpha. \end{aligned}$$

Next we consider any $i < \ell - 1$. We wish to show

$$(16) \quad \begin{aligned} u_{i, i} c_i + u_{i, i+1} c_{i+1} + \dots + u_{i, \ell-1} c_{\ell-1} + u_{i, \ell} c_\ell &= \alpha \\ \Leftrightarrow u_{i, i} c_i + \sum_{p=1}^{\ell-1-i} u_{i, i+p} c_{i+p} + u_{i, \ell} c_\ell &= \alpha \\ \Leftrightarrow u_{i, i} c_i + \sum_{p=1}^{\ell-1-i} u_{i, i+p} c_{i+p} &= \alpha - u_{i, \ell} c_\ell \\ \Leftrightarrow u_{i, i} c_i + \sum_{p=1}^{\ell-1-i} u_{i, i+p} c_{i+p} &= \alpha^2, \end{aligned}$$

where we have used our computation of $u_{\ell, \ell} c_\ell$ in (15) above and the definition of α to obtain the last line (16).

Now we substitute $u_{i,j}$ and c_j into (16):

$$\begin{aligned}
& \frac{2(f-i+2)}{f+i} \alpha \frac{(f+i)(f-\ell+1)}{(f-i)(f-i+1)(f-i+2)} + \sum_{p=1}^{\ell-1-i} \frac{4}{f+i+p} \alpha \frac{(f+i+p)(f-\ell+1)}{(f-i-p)(f-i-p+1)(f-i-p+2)} = \alpha^2 \\
\Leftrightarrow & \frac{2(f-\ell+1)}{(f-i)(f-i+1)} + \sum_{p=1}^{\ell-1-i} \frac{4(f-\ell+1)}{(f-i-p)(f-i-p+1)(f-i-p+2)} = \frac{2}{f-\ell+2} \\
\Leftrightarrow & \frac{1}{(f-i)(f-i+1)} + \sum_{p=1}^{\ell-1-i} \frac{2}{(f-i-p)(f-i-p+1)(f-i-p+2)} = \frac{1}{(f-\ell+1)(f-\ell+2)} \\
(17) \Leftrightarrow & \sum_{p=1}^{\ell-1-i} \frac{2}{(f-i-p)(f-i-p+1)(f-i-p+2)} = \frac{1}{2} \left(\frac{1}{(f-\ell+1)(f-\ell+2)} - \frac{1}{(f-i)(f-i+1)} \right).
\end{aligned}$$

We substitute $y = f - i$ and $m = \ell - 1 - i$. Then (17) becomes

$$\sum_{p=1}^m \frac{1}{(y-p)(y-p+1)(y-p+2)} = \frac{1}{2} \left(\frac{1}{(y-m)(y-m+1)} - \frac{1}{y(y+1)} \right),$$

which reduces to the identity (14). □

We have now proved all the lemmas used in the proof of Proposition 3.4.

3.2.1. *Example:* $n = 9$. Here $f = 4$. The four critical α are $2/5$ (which corresponds to $(\mathbb{P}^1)^9 // \text{SL}(2)$), $1/2$, $2/3$, and 1 (which corresponds to $\overline{M}_{0,9}$). We use the three vital curves $C(4, 3, 1, 1)$, $C(5, 2, 1, 1)$, and $C(6, 1, 1, 1)$ and the line bundles $V(1/5, 9)$, $V(1/6, 9)$, $V(1/7, 9)$ to form the intersection matrix:

$$\begin{array}{cccc}
& & V(1/5, 9) & V(1/6, 9) & V(1/7, 9) \\
C(4, 3, 1, 1) & & 2 & 2/3 & 4/7 \\
C(5, 2, 1, 1) & & 0 & 4/3 & 4/7 \\
C(6, 1, 1, 1) & & 0 & 0 & 6/7
\end{array},$$

and the four vectors $C_i \cdot A_\alpha$ are

$$\begin{array}{cccc}
\alpha : & 2/5 & 1/2 & 2/3 & 1 \\
C(4, 3, 1, 1) \cdot A_\alpha : & 3/5 & 1/2 & 2/3 & 1 \\
C(5, 2, 1, 1) \cdot A_\alpha : & 0 & 1/2 & 2/3 & 1 \\
C(6, 1, 1, 1) \cdot A_\alpha : & 0 & 0 & 1/3 & 1
\end{array}.$$

This leads to the following equivalences:

$$\begin{aligned}
A_{2/5} & \equiv \frac{3}{10} V(1/5, 9) \\
A_{1/2} & \equiv \frac{1}{8} V(1/5, 9) + \frac{3}{8} V(1/6, 9) \\
A_{2/3} & \equiv \frac{1}{9} V(1/5, 9) + \frac{1}{3} V(1/6, 9) + \frac{7}{18} V(1/7, 9) \\
A_1 & \equiv \frac{1}{12} V(1/5, 9) + \frac{1}{4} V(1/6, 9) + \frac{7}{6} V(1/7, 9).
\end{aligned}$$

3.3. When n is even. We obtain similar results by similar methods.

Once again write $f = \lfloor n/2 \rfloor = n/2$ since n is even. We use the curves $C_i = C(f-2+i, f-i, 1, 1)$ for $i = 1, \dots, f-1$ as our basis for the intersection matrix. The resulting system is

$$(18) \quad \begin{pmatrix} 4 & \frac{4}{f+1} & \frac{4}{f+2} & \cdots & \frac{4}{f+j-1} & \cdots & \frac{4}{2f-2} \\ 0 & \frac{2f}{f+1} & \frac{4}{f+2} & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \frac{2(f-1)}{f+2} & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & 0 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 0 & \frac{2(f-j+2)}{f+j-1} & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \frac{4}{2f-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{2f-2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_{f-1} \end{pmatrix} = \begin{pmatrix} \alpha \\ \vdots \\ \alpha \\ 1-\alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and it has the solution given below. Note that more formulas are required when n is even than when n is odd. We may attribute this to the fact that C_1 and $V(1/n, n)$ are more symmetric than any other pair under consideration, and as a result $u_{1,1}$ does not fit the pattern observed in the other diagonal entries of the intersection matrix when n is even.

Proposition 3.10. *Suppose $n \geq 6$ is even.*

If $\alpha = \frac{2}{f+1}$ (which corresponds to $(\mathbb{P}^1)^n // \mathrm{SL}(2)$), then

$$A_\alpha \equiv (f-2)/(2f+2)V(1/f, n).$$

If $\alpha = \frac{2}{f}$, then

$$A_\alpha \equiv \frac{1}{f^2}V\left(\frac{1}{f}, n\right) + \frac{(f+1)(f-2)}{2f^2}V\left(\frac{1}{f+1}, n\right).$$

If $\alpha = \frac{2}{f-\ell+2}$ for some $\ell \in \{3, \dots, f-1\}$, then

$$A_\alpha \equiv c_1V\left(\frac{1}{f}, n\right) + \cdots + c_{f-1}V\left(\frac{1}{2f-1}, n\right),$$

where

$$(19) \quad \begin{aligned} c_1 &= \frac{1}{2}\alpha \frac{f-\ell+1}{f(f-1)}, \\ c_i &= \alpha \frac{(f-\ell+1)(f-1+i)}{(f-i)(f-i+1)(f-i+2)} \text{ if } 2 \leq i \leq \ell-1, \\ c_\ell &= \frac{1}{4}\alpha \frac{(f-\ell)(f+\ell-1)}{f-\ell+2}, \\ c_i &= 0 \text{ if } i > \ell. \end{aligned}$$

If $\alpha = 1$ and $n = 6$ (which corresponds to $\overline{M}_{0,6}$), then

$$A_\alpha \equiv \frac{1}{12}V\left(\frac{1}{3}, 6\right) + \frac{2}{3}V\left(\frac{1}{4}, 6\right).$$

If $\alpha = 1$ and $n \geq 8$ (which corresponds to $\overline{M}_{0,n}$), then

$$A_\alpha \equiv c_1V\left(\frac{1}{f}, n\right) + \cdots + c_{f-1}V\left(\frac{1}{2f-1}, n\right)$$

where

$$(20) \quad \begin{aligned} c_1 &= \frac{1}{2f(f-1)}, \\ c_i &= \frac{(f-1+i)}{(f-i)(f-i+1)(f-i+2)} \text{ if } 2 \leq i \leq f-2, \\ c_{f-1} &= \frac{f-1}{3}. \end{aligned}$$

3.4. Nefness of A_α . Since each critical A_α is an effective combination of the $L_{\vec{x}}$, and every A_α is a convex combination of the critical A_α , we have the following corollary:

Corollary 3.11. A_α is nef for all $\frac{4}{n+1} \leq \alpha \leq 1$.

4. AMPLENESS OF A_α ON $\overline{M}_{0,\beta}$

We use the following result of [Ale08] to prove that A_α is ample on $\overline{M}_{0,\beta}$. Below, $\beta = (b_1, \dots, b_n)$, and $\Delta_\beta(2, n)$ denotes the weighted hypersimplex, the subset of \mathbb{R}^n given by the equality $\sum_{i=1}^n x_i = 2$ and the inequalities $0 \leq x_i \leq b_i$.

Theorem 4.1. Let $\{\vec{x}_i\}$ be a finite set of points of $\Delta_\beta(2, n)$ satisfying the following conditions:

- (1) Each $L_{\vec{x}_i}$ is the pullback of a \mathbb{Q} -line bundle from $\overline{M}_{0,\beta}$.
- (2) For every matroid polytope P with $(\text{Int } P) \cap \Delta_\beta(2, n) \neq \emptyset$, there exists an \vec{x}_i in $\text{Int } P$.

Then the sheaf $\sum L_{\vec{x}_i}$ on $\overline{M}_{0,n}$ is the pullback of an ample \mathbb{Q} -line bundle from $\overline{M}_{0,\beta}$.

$\overline{M}_{0,1/k, \dots, 1/k}$ corresponds to the critical alpha value $\alpha = 2/(k+1)$, and we write A_α as an effective combination of $V(1/(\lfloor n/2 \rfloor + 1, n), \dots, V(1/(n-k), n)$.

Proposition 4.2. Let $k \in \{1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$. Write $\beta = (1/k, \dots, 1/k)$, $f = \lfloor n/2 \rfloor$, and $\ell = f+1-k$ as in the previous section. Then any combination $c_1 V(1/f, n) + \dots + c_\ell V(1/(f+\ell), n) = c_1 V(1/f, n) + \dots + c_\ell V(1/(n-k), n)$ with all the coefficients $c_i > 0$ is ample on $\overline{M}_{0,\beta}$. In particular, when $\alpha = 2/(k+1)$, A_α is ample on $\overline{M}_{0,\beta}$.

Proof. Suppose P_Λ is a matroid polytope. We say P_Λ is *needed* or *necessary* if $(\text{Int } P_\Lambda) \cap \Delta_\beta(2, n) \neq \emptyset$. We say that a necessary P_Λ is *captured* if there is an $\vec{x} \in (\text{Int } P_\Lambda) \cap \Delta_\beta(2, n)$ such that $L_{\vec{x}}$ appears in $c_1 V(1/f, n) + \dots + c_\ell V(1/(n-k), n)$. Now we show that the above line bundles capture all necessary matroid polytopes, and then apply Theorem 4.1.

Reduction 1. It is enough to consider partitions Λ of width 3. Every necessary matroid polytope contains a necessary matroid polytope of partition width 3: Suppose $\Lambda = I_1 \amalg \dots \amalg I_w$ with $w \geq 4$. Recall that

$$P_\Lambda := \{\vec{x} \in \Delta(2, n) \mid x_{I_1} \leq 1, \dots, x_{I_w} \leq 1\}.$$

By assumption, P_Λ is necessary, so there is some $\vec{x} \in (\text{Int } P_\Lambda) \cap \Delta_\beta(2, n)$ such that $x_{I_j} \leq 1$ for all $j = 1, \dots, w$. Reorder I_1, \dots, I_w so that $x_{I_1} \geq \dots \geq x_{I_w}$. Define j_2 to be the index such that

$$(21) \quad x_{I_{j_2}} + \dots + x_{I_w} < 1$$

while

$$(22) \quad x_{I_{j_2-1}} + x_{I_{j_2}} + \dots + x_{I_w} \geq 1.$$

Define j_1 to be the index such that

$$(23) \quad x_{I_{j_1}} + \dots + x_{I_{j_2-1}} < 1$$

while

$$(24) \quad x_{I_{j_1-1}} + x_{I_{j_1}} + \dots + x_{I_{j_2-1}} \geq 1.$$

Then P_Λ contains the matroid polytope $P_{\Lambda'}$, where

$$\Lambda' = (I_1 \amalg \cdots \amalg I_{j_1-1}) \amalg (I_{j_1} \amalg \cdots \amalg I_{j_2-1}) \amalg (I_{j_2} \amalg \cdots \amalg I_w).$$

Moreover, we will show that $P_{\Lambda'}$ contains \vec{x} in its interior, so it is a necessary matroid polytope of partition width 3.

We show $\vec{x} \in \text{Int } P_{\Lambda'}$: By construction, $x_{I_{j_1}} + \cdots + x_{I_{j_2-1}} < 1$ and $x_{I_{j_2}} + \cdots + x_{I_w} < 1$. So it remains only to show that $x_{I_1} + \cdots + x_{I_{j_1-1}} < 1$. For this, we know

$$(25) \quad x_{I_1} + \cdots + x_{I_{j_1-1}} + x_{I_{j_1}} + \cdots + x_{I_{j_2-1}} + x_{I_{j_2}} + \cdots + x_{I_w} = 2$$

since $\vec{x} \in \Delta_\beta(2, n)$. Adding the inequalities (22) and (24) and combining the result with (25), we obtain

$$(26) \quad x_{I_1} + \cdots + x_{I_{j_1-1}} \leq x_{I_{j_2-1}}.$$

Since $\vec{x} \in \text{Int } P_\Lambda$, we have $x_{I_{j_2-1}} < 1$.

(We remark further that $j_1 = 2$ for most \vec{x} and Λ . Otherwise, if $j_1 > 2$, then (26) combined with $x_{I_{j_1-2}} \geq x_{I_{j_2-1}}$ implies that we must have $j_1 = 3$ and $x_{I_1} = x_{I_2} = \cdots = x_{I_{j_2-1}}$, which will only happen for very special \vec{x} and Λ .)

This completes our first reduction step. It allows us to conclude that if all the necessary matroid polytopes for partitions of width 3 are captured, then all the matroid polytopes are captured. So assume from now on that $w = 3$.

Reduction 2. We may assume without loss of generality that $n_1 \geq n_2 \geq n_3$. Recall that by definition $V(a, n)$ is a tensor product of the n line bundles of the form $L_{\vec{x}}$ where \vec{x} has $n - 1$ entries equal to a and one entry equal to \bar{a} . For a given matroid polytope P_Λ , it is enough to show that one of these $L_{\vec{x}}$ captures it, and so by symmetry we may assume $n_1 \geq n_2 \geq n_3$.

Reduction 3. It's enough to consider $k \geq 2$. There are two reasons for this. First, $\overline{M}_{0,(1)^n} \cong \overline{M}_{0,(2/3)^n}$, and the second space corresponds to $k = 2$. Second, $\overline{M}_{0,(1)^n} \cong \overline{M}_{0,n}$, and it is well-known that $K + \Delta$ is ample on $\overline{M}_{0,n}$.

With these reductions, this proposition (Proposition 4.2) now follows from the next proposition (Proposition 4.3, below). \square

Proposition 4.3. *Suppose $2 \leq k \leq \lfloor n/2 \rfloor - 1$. Let $\beta = (1/k, \dots, 1/k)$. Suppose $\Lambda = (n_1, n_2, n_3)$ is a partition of n into three nonzero parts such that $(\text{Int } P_\Lambda) \cap \Delta_\beta(2, n) \neq \emptyset$. Assume $n_1 \geq n_2 \geq n_3$.*

- a. *If $n_3 \geq k$, then $L(1/(n-1-n_3))$ captures P_Λ .*
- b. *If $n_3 < k$, then $L(1/(n-k))$ captures P_Λ .*

Proof. First, we study the intersection $(\text{Int } P_\Lambda) \cap \Delta_\beta(2, n)$. If

$$(27) \quad \min_{\vec{x} \in \Delta_\beta(2, n)} x_{I_1} = 2 - \max_{\vec{x} \in \Delta_\beta(2, n)} (x_{I_2} + x_{I_3}) \geq 1$$

then $(\text{Int } P_\Lambda) \cap \Delta_\beta(2, n) = \emptyset$. Here, $\beta = (1/k, \dots, 1/k)$. Therefore, (27) becomes

$$(28) \quad \begin{aligned} 2 - (n_2k + n_3k) &\geq 1 \\ \Leftrightarrow k &\geq n_2 + n_3 \end{aligned}$$

So the hypothesis that $(\text{Int } P_\Lambda) \cap \Delta_\beta(2, n) \neq \emptyset$ implies that $n_2 + n_3 > k$. Then we also have $n_1 < n - k$.

Now we prove part a. We need to check that $x_{I_j} < 1$ for $L(1/(n-1-n_3))$ for $j = 1, 2, 3$. First we check x_{I_1} . This is

$$\begin{aligned} \frac{n_1}{n-1-n_3} &< 1 \\ \Leftrightarrow 1 &< n - n_1 - n_3 = n_2. \end{aligned}$$

If $n_2 = 1$, then $n_3 = 1$ also, since $n_2 \geq n_3$. But then $k < n_2 + n_3 = 2$, which contradicts the hypothesis that $k \geq 2$. Therefore, $n_2 > 1$ as desired.

For $j = 2$, we have $x_{I_2} \leq x_{I_1} < 1$ by the previous paragraph.

For $j = 3$, we have

$$x_{I_3} = \left(2 - (n-1) \frac{1}{n-1-n_3} \right) + (n_3-1) \frac{1}{n-1-n_3} < 1$$

$$\Leftrightarrow -1 < 0.$$

This completes part a.

Now we prove part b. $n_1 < n - k$, so $x_{I_1} = \frac{n_1}{n-k} < 1$. Also, since $n_2 \leq n_1$, we have $x_{I_2} \leq x_{I_1} < 1$. Finally,

$$x_{I_3} = \left(2 - (n-1) \frac{1}{n-k} \right) + (n_3-1) \frac{1}{n-k} = \frac{n+n_3-2k}{n-k},$$

and this is less than 1, since by hypothesis $n_3 < k$ in part b. \square

Proposition 4.2 together with [Sim08] Corollary 2.3.5 now yields the following result, which has also recently been proved by Fedorchuk and Smyth.

Corollary 4.4 (extending [Sim08] Theorem 2.4.5; cf. [FS08]). *Fix $n \geq 4$ and α a rational number in $(\frac{2}{n-1}, 1]$. Let $\overline{M}_{0,n}(\alpha)$ denote the log canonical model of $\overline{M}_{0,n}$ with respect to $K + \alpha\Delta$.*

If α is in the range $(\frac{2}{k+2}, \frac{2}{k+1}]$ for some $k = 1, \dots, \lfloor \frac{n-1}{2} \rfloor$, then $\overline{M}_{0,n}(\alpha) \cong \overline{M}_{0,(1/k, \dots, 1/k)}$. If α is in the range $(\frac{2}{n-1}, \frac{2}{\lfloor \frac{n}{2} \rfloor + 1}]$, then $\overline{M}_{0,n}(\alpha) \cong (\mathbb{P}^1)^n // \text{SL}(2)$, where the linearization is given by symmetric weights.

4.1. The $V(a, n)$ do not always give $\overline{M}_{0,\beta}$'s. The $V(a, n)$ are big and nef, and the previous subsection shows that effective combinations of sufficiently many of them are ample on certain $\overline{M}_{0,\beta}$. Here we present an example to show that a single $V(a, n)$ taken by itself may not be ample on any $\overline{M}_{0,\beta}$.

Proposition 4.5. *Suppose that S is a semiample S_n -equivariant line bundle on $\overline{M}_{0,n}$, and that the image of $\overline{M}_{0,n}$ under the linear system $|S|$ is isomorphic to a weighted moduli space $\overline{M}_{0,\beta}$. Then it is also isomorphic to a weighted moduli space $\overline{M}_{0,\beta'}$, where β' is a symmetric set of weights.*

Proof. Recall from [Has03] that the space of weights β has a chamber decomposition, again given by hyperplanes of the form $\sum_{i \in I, \#I \geq 3, \#I^c \geq 3} \beta_i = 1$. If two sets of weights lie in the interior of the same chamber, then the resulting moduli spaces are isomorphic.

First note that if the moduli spaces $\overline{M}_{0,\sigma\beta}$ all lie in a single chamber, then $\overline{M}_{0,\beta'}$, where β' is the average of all the $\sigma\beta$, lies in this chamber too.

Now suppose that $\overline{M}_{0,\beta}$ and $\overline{M}_{0,\sigma\beta}$ lie in two different chambers. There is no isomorphism between $\overline{M}_{0,\beta}$ and $\overline{M}_{0,\sigma\beta}$ commuting with π_β and $\pi_{\sigma\beta}$, so this contradicts S being symmetric. To see that no such isomorphism is possible, suppose that β satisfies $\sum_{i \in I} \beta_i \leq 1$ while $\sum_{i \in I} \sigma\beta_i > 1$, and consider the locus T in $\overline{M}_{0,n}$ whose generic point is a curve with two components: the first component contains the points labelled by I , and these and the point of attachment are allowed to vary; the second component contains the points of I^c , which are fixed along with the point of attachment. Then T is isomorphic to $\overline{M}_{0,|I|+1}$, and has dimension $|I| - 2 \geq 1$. But $\pi_\beta(T)$ is a point, while $\pi_{\sigma\beta}(T) \cong T$. \square

Now we consider a specific example: the line bundle $V(1/6, 8)$ when $n = 8$. We compute the intersections of $V(1/6, 8)$ and the critical A_α with all the vital curves:

	A_1	$A_{1/2}$	$A_{2/5}$	$V(1/6, 8)$
$C(3, 3, 1, 1)$	1	1/2	4/5	2/3
$C(4, 2, 1, 1)$	1	1/2	0	2/3
$C(5, 1, 1, 1)$	1	0	0	1
$C(3, 2, 2, 1)$	1	1	4/5	1/3
$C(2, 2, 2, 2)$	1	3/2	8/5	0

From this data it is clear that the contraction defined by $V(1/6, 8)$ does not give any $\overline{M}_{0,\beta}$. If the image were any $\overline{M}_{0,\beta}$, then by the proposition, it would be possible to use a symmetric β . But we know the discrepancies for all symmetric β : these are precisely the A_α . $V(1/6, 8)$ is zero on $[C(2, 2, 2, 2)]$, but none of the A_α are zero on this class.

CONCLUDING REMARKS

In this note we have used only a small number of the $L_{\vec{x}}$. Many interesting questions remain, including:

- (1) Can we identify log canonical models of $\overline{M}_{0,n}$ for nonsymmetric divisors as moduli spaces $\overline{M}_{0,\beta}$ where β is a nonsymmetric set of weights?
- (2) Is there a combinatorial interpretation to our coefficients c_i ? Do they count something? Or, can we get a counting proof if we also keep track of stabilizers and the number of vital curves having a given partition shape?
- (3) Can we write all symmetric F-nef divisors as effective combinations of the $L_{\vec{x}}$ and boundary divisor classes? Does using these line bundles make it easier to check the S_n -equivariant F-conjecture?

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