

TUTORIAL: THE EICHLER TRACE FORMULA FOR REPRESENTATIONS OF AUTOMORPHISMS OF CURVES

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Suppose C is a (connected, reduced, projective) smooth algebraic curve. Then the automorphisms of C act on $H^0(C, K)$. This action has been much studied, going back to at least the mid-1800s.

I became interested in representations of automorphisms of curves for their possible applications in proving GIT stability. One of the main tools in the subject is the Eichler trace formula, and I worked this out for some very basic examples. It was a great exercise for me, and maybe now it can be a great exercise for you, too!

0.1. Wiman curves. A theorem due to Wiman states that the maximum order of an automorphism of a curve is $4g + 2$. There is a unique hyperelliptic curve in each genus $g \geq 3$ whose full automorphism group is the cyclic group of this order (it is the curve given by the equation $y^2 = x^{2g+1} - 1$); Kulkarni ([Kul]) calls these the *Wiman curves of type I*.

I noticed that the representations of $\text{Aut}(C)$ for the Wiman curves of type I had all distinct characters (just stare at the usual basis of differentials for a hyperelliptic curve). So I decided to try the calculation in genus 3 first to gain practice using the formula, since in genus 3 the calculation would be fairly small. Upon completing the calculation, I realized that every step generalized immediately to any g for which $p = 2g + 1$ is prime. This yields the following result.

Proposition 1. *For any $g \geq 3$ for which $p := 2g + 1$ is prime, let C_g be the Wiman curve of type I in genus g . Then the representation of $\text{Aut}(C_g)$ on $H^0(C_g, K)$ is isomorphic to the following representation:*

$$(1) \quad G \xrightarrow{\rho} GL(g, \mathbb{C})$$

$$(2) \quad [1] \mapsto A := \begin{pmatrix} \zeta_{2p} & & & & \\ & \zeta_{2p}^3 & & & \\ & & \zeta_{2p}^5 & & \\ & & & \ddots & \\ & & & & \zeta_{2p}^{2g-1} \end{pmatrix}.$$

In particular the characters are all distinct.

I will divide the proof into three steps:

- (1) Describe the background/setup for using the Eichler trace formula;
- (2) Compute the character χ_E of the representation of $\text{Aut}(C_g)$ using the Eichler trace formula;
- (3) Show that this matches the character χ_ρ of the representation generated by the matrix A in (2).

0.1.1. *Background and setup for using the Eichler trace formula.* The order of $\text{Aut}(C_g)$ is $4g + 2 = 2p$. Let G be the cyclic group of order $2p$; I will identify G with $\mathbb{Z}/2p\mathbb{Z}$, denoting elements of G as $[1], [2], \dots$.

We need something called a *surface kernel epimorphism* $\Phi : \Gamma \rightarrow G$. The condition of being surface kernel ensures that G is (a subgroup of) an automorphism group of a curve.

Proposition 2 ([Br], 9.7 and 3.16). *Write $m_1 = 2$, $m_2 = p$, and $m_3 = 2p$. Define the Fuchsian group Γ as follows:*

$$(3) \quad \Gamma := \langle a_1, b_1, \dots, a_g, b_g, c_1, c_2, c_3 \mid c_1^{m_1}, c_2^{m_2}, c_3^{m_3}, \prod_{i=1}^g [a_i, b_i] \cdot c_1 c_2 c_3 \rangle.$$

Then the map $\Phi : \Gamma \rightarrow G$ defined by

$$(4) \quad \begin{aligned} a_i &\mapsto [0] \\ b_i &\mapsto [0] \\ c_1 &\mapsto [k_1] \\ c_2 &\mapsto [k_2] \\ c_3 &\mapsto [k_3] \end{aligned}$$

is a surface kernel epimorphism if the following conditions are satisfied:

- (1) $\sum k_i \equiv 0 \pmod{2p}$
- (2) $m \mid m_i k_i$ for $i = 1, 2, 3$
- (3) $m_i = 2p / \gcd(m, k_i)$ for $i = 1, 2, 3$.

We take $k_1 = p$, $k_2 = p - 1$, and $k_3 = 1$, which fit the requirements of Proposition 2. The effect of these choices is discussed in the remarks following the proof.

Finally for any positive integer m , write $I(m)$ for the set of positive integers smaller than m that are coprime to m . That is,

$$(5) \quad I(m) := \{u \mid 1 \leq u < m, \gcd(u, m) = 1\}.$$

The next result tells us how to obtain the character of the representation of $\text{Aut}(C_g)$ on $H^0(C_g, K)$ from Φ .

Proposition 3 (Eichler trace formula). *Let $[h] \in G, h \neq [0]$. Write m for the order of $[h]$. Let $u \in I(m)$. Then the character χ_E of the representation of G on $H^0(C, K)$ is given as follows:*

$$(6) \quad \chi_E([h]) = 1 + \sum_{u \in I(m)} |Fix_u(h)| \frac{\zeta_m^u}{1 - \zeta_m^u},$$

where

$$(7) \quad |Fix_u(h)| = |Cent_G(h)| \cdot \sum_{\substack{m \mid m_i \\ h \sim \Phi(c_i)^{m_i u/m}}} \frac{1}{m_i}$$

Proof. This is a combination of Theorem 12.1 in [Br] and Lemma 11.5 in [Br]. Breuer in turn cites [Harv] Theorem 7 and [FK] V.2.4–V.2.9.

0.1.2. *Using the Eichler trace formula to compute χ_E .* It is now fairly straightforward to compute the value of χ_E on each conjugacy class of G . Since G is abelian, each conjugacy class contains exactly one element. We will split up our calculation according to the order of each element. Also, note that since G is abelian, the centralizer of any element is the whole group G , so the term $|\text{Cent}_G(h)|$ in (7) is $2p$ for all $h \in G$.

For the identity element, $\chi_E([0]) = g$ since $\dim H^0(C, K) = g$.

For the element of order 2 (that is, the class $[p]$), we have $I(2) = \{1\}$. Then Proposition (3) reads

$$\chi_E([p]) = 1 + |\text{Fix}_1([p])| \frac{\zeta_2}{1 - \zeta_2}.$$

There are two terms in the summation in (7): we have $2|2$ and $[p] \sim 1 \cdot [p]$, and we have $2|2p$ and $[p] \sim p \cdot [1]$. Thus

$$|\text{Fix}_1([p])| = 2p \left(\frac{1}{2} + \frac{1}{2p} \right) = p + 1$$

and

$$\chi_E([p]) = 1 + (p + 1) \left(\frac{-1}{2} \right) = -g.$$

Next we consider the elements of order p (that is, the classes $[j]$ where j is even and nonzero). The set $I(p) = \{1, \dots, p-1\}$. It is not too difficult to compute $|\text{Fix}_u([j])|$ for the j 's and u 's that can occur. The summation in (7) can have at most two terms: $p|p$ and we will add $\frac{1}{m_2} = \frac{1}{p}$ if $[j] \sim u[p-1]$, and $p|2p$ and we add $\frac{1}{m_3} = \frac{1}{2p}$ if $[j] \sim 2u[1]$. Thus

$$(8) \quad \text{Fix}_u([j]) = \begin{cases} 2 & \text{if } j \equiv u(p-1) \pmod{2p} \\ 1 & \text{if } u = j/2 \\ 0 & \text{otherwise} \end{cases}$$

This leads to

$$(9) \quad \chi_E([j]) = 1 + \frac{\zeta_p^{j/2}}{1 - \zeta_p^{j/2}} + 2 \frac{\zeta_p^{-j}}{1 - \zeta_p^{-j}}.$$

By writing the terms in the RHS of (9) over a common denominator and then canceling factors, we obtain a simpler expression:

$$(10) \quad \chi_E([j]) = \frac{-1}{1 + \zeta_{2p}^j}.$$

Finally we consider the elements of order $2p$ (that is, the classes $[k]$ where k is odd and not equal to p). The set $I(2p)$ is also the set of odd integers not equal to p . The summation in (7) can have at most one term: $2p|2p$ and we will add $\frac{1}{m_3} = \frac{1}{2p}$ if $[k] \sim u[1]$. Thus

$$(11) \quad \text{Fix}_u([k]) = \begin{cases} 1 & \text{if } u = k \\ 0 & \text{otherwise} \end{cases}$$

Then

$$(12) \quad \chi_E([k]) = 1 + \frac{\zeta_{2p}^k}{1 - \zeta_{2p}^k} = \frac{1}{1 - \zeta_{2p}^k}.$$

0.1.3. *The characters χ_E and χ_ρ are equal.* Having computed χ_E in the previous subsection, we now complete the proof of Proposition 1 by showing that the character χ_ρ associated to the representation generated by the matrix A in (2) is equal to the character χ_E . Once again, we check this on each conjugacy class according to their orders.

First, we record the values of χ_ρ for reference:

$$\chi_\rho([i]) = \sum_{n=1}^g \zeta_{2p}^{i(2n-1)}.$$

Clearly for the identity element, χ_E and χ_ρ match.

For the element of order 2, we compute $\chi_\rho([p]) = \sum_{n=1}^g \zeta_{2p}^{p(2n-1)} = g \cdot (-1) = -g$, which matches $\chi_E([p])$.

For an element $[j]$ of order p , the equality of $\chi_E([j])$ and $\chi_\rho([j])$ is expressed by the following identity:

$$(13) \quad \zeta_{2p}^j + \zeta_{2p}^{3j} + \cdots + \zeta_{2p}^{(p-2)j} = \frac{-1}{1 + \zeta_{2p}^j}.$$

Note that the subscripts and exponents are all even. Write $\ell = j/2$. We reduce both the subscripts and exponents by a factor of 2 and multiply on both sides by $(1 + \zeta_p^\ell)$. Then the left hand side becomes

$$\zeta_p^\ell + \zeta_p^{3\ell} + \cdots + \zeta_p^{(p-2)\ell} + \zeta_p^{2\ell} + \zeta_p^{4\ell} + \cdots + \zeta_p^{(p-1)\ell}.$$

Here ℓ and p are coprime, so in some order this is just the sum of the p^{th} roots of unity with 1 omitted. Hence, this is -1 , and the identity (13) is true, as required.

Finally, for the elements $[k]$ of order $2p$, the equality of $\chi_E([j])$ and $\chi_\rho([j])$ is expressed by the following identity:

$$(14) \quad \zeta_{2p}^k + \zeta_{2p}^{3k} + \cdots + \zeta_{2p}^{(p-2)k} = \frac{1}{1 - \zeta_{2p}^k}.$$

Multiply both sides by $(1 - \zeta_{2p}^k)$, and rewrite $-1 = \zeta_{2p}^p$. Then the desired identity takes the form

$$\zeta_{2p}^k + \zeta_{2p}^{3k} + \cdots + \zeta_{2p}^{(p-2)k} + \zeta_{2p}^{(p+2)k} + \zeta_{2p}^{(p+4)k} + \cdots + \zeta_{2p}^{(2p-1)k} = 1.$$

i.e. the left hand side is all the odd powers of ζ_{2p}^k except ζ_{2p}^{pk} . Now, since p and k are coprime, in some order the left hand side is just the sum of the odd power roots of unity (except the p^{th} one), and the identity we want to prove is:

$$\zeta_{2p} + \zeta_{2p}^3 + \cdots + \zeta_{2p}^{(p-2)} + \zeta_{2p}^{(p+2)} + \zeta_{2p}^{(p+4)} + \cdots + \zeta_{2p}^{(2p-1)} = 1.$$

Now, add $\zeta_{2p}^2 + \zeta_{2p}^4 + \cdots + \zeta_{2p}^{2p-2}$ to both sides. On the right hand side we get

$$1 + \zeta_{2p}^2 + \zeta_{2p}^4 + \cdots + \zeta_{2p}^{2p-2} = 1 + \zeta_p + \zeta_p^2 + \cdots + \zeta_p^{p-1} = 0,$$

and on the left hand side we also have zero, since we have the sum of all the powers of $(2p)^{\text{th}}$ roots of unity except ζ_{2p}^0 and ζ_{2p}^p . Thus the identity (14) is true, as required.

This shows that $\chi_E = \chi_\rho$ and completes the proof of Proposition 1.

0.1.4. *Remarks.*

- (1) In the proof I chose $k_1 = p$ and $k_2 = p - 1$ and $k_3 = 1$. The choice of k_1 is forced, since there is only one element of order 2. But there is nothing special about my choice of k_2 and k_3 ; we could have chosen k_3 to be any odd number not equal to p , and then k_2 is determined by the condition $k_1 + k_2 + k_3 \equiv 0 \pmod{2p}$. The rest of the calculation goes through with minor changes, and we obtain the character of the representation $\rho' : [k_3] \mapsto A$. This certainly gives another multiplicity free representation; however, I have forgotten the definition of when representations are isomorphic, and I can't remember if this is considered isomorphic to the representation ρ .
- (2) The hypothesis $p = 2g + 1$ permits a very easy proof, but it is unnecessary—the conclusion is true for any g . Someone with greater number theory facility might be able to give a proof using these techniques.
- (3) Many of the theorems in the subject are of the form: “let G be a subgroup of $\text{Aut}(X)$ for some X . Then . . .” The Eichler trace formula is one of them. So, in theory, it appears that we have possibly proven something about a larger set of curves than just the Wiman curves, i.e. whenever the cyclic group G of order $2p$ appears as a subgroup of $\text{Aut}(X)$ and X has genus g , then the representation of G on $H^0(X, K)$ is what we calculated in Proposition 1. Actually, this is an illusion; the only smooth curves of genus g with an automorphism of order $4g + 2$ are the Wiman curves of type I. See [Kul], Section 7.

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