Nef line bundles on $\overline{M}_{0,n}$ from GIT

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Many of the results here are joint work with Valery Alexeev.

We have a preprint: arXiv:0812.0778

I will also discuss work by Najmuddin Fakhruddin, and some very recent results by the UGA VIGRE Algebraic Geometry group: Angela Gibney, Valery Alexeev, David Swinarski, Maxim Arap, Lev Konstantinovskiy, Jim Stankewicz, and Boris Alexeev.

Moduli spaces of curves

Recall: M_g is the moduli space of smooth curves of genus g.

There is a nice compactification $\overline{M}_g$; the boundary consists of nodal curves with finite automorphism groups (Deligne-Mumford stable curves)

$\overline{M}_{g,n}$: moduli space of nodal curves with n distinct ordered smooth points and finite automorphism group

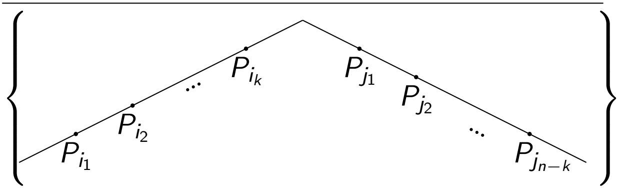
An important special case is when $g = 0$. The geometry of $\overline{M}_{0,n}$ is very combinatorial and very intricate.

Divisors on $\overline{M}_{0,n}$

Write $\Delta := \overline{M}_{0,n} \setminus M_{0,n}$ for the boundary divisor.

Irreducible components of $\Delta \leftrightarrow$ partitions of $\{1, \ldots, n\}$ into two subsets I and J of size at least 2.

$\Delta_{I,J} =$



The boundary divisor classes $\delta_{I,J} := [\Delta_{I,J}]$ span $\mathrm{Pic}(\overline{M}_{0,n})$. The relations between them are called the Keel relations.

The symmetric group S_n acts on $\overline{M}_{0,n}$ by permuting the marked points.

We will often consider symmetric divisors. There is a popular basis of $\mathrm{Pic}(\overline{M}_{0,n})^{S_n}$ given by $B_2, \ldots, B_{\lfloor n/2 \rfloor}$, where

$$B_k = \sum_{I \subset \{1,\ldots,n\} : \# I = k} \delta_{I,J}.$$

The dual graph of a curve

Let C be a nodal curve with marked points $\{P_i\}$.

DEFINITION The dual graph G_C is the following graph:

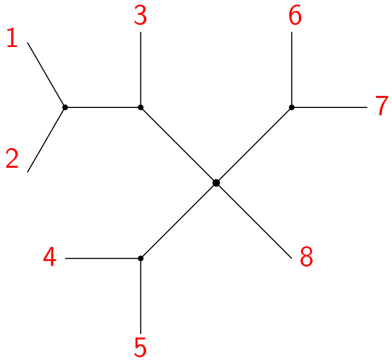
vertices of G_C : irreducible components of C

edges of G_C : for each node between components C_i and C_j, there is an edge between $v(C_i)$ and $v(C_j)$
leaves of G_C : if a marked point P_i lies on component C_j then the leaf labelled i is attached to $v(C_j)$.

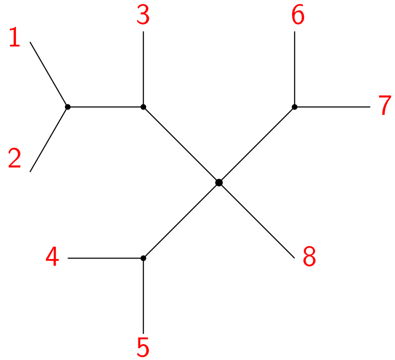
F-curves

Let G be a tree with n labelled leaves.

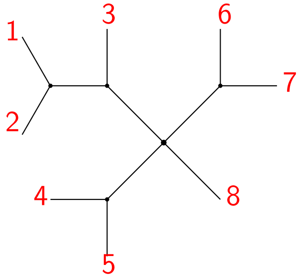
Suppose that G is trivalent at every vertex except one, where it is 4-valent.



Definition An F-curve in $\overline{M}_{0,n}$ corresponding to G is the locus of curves whose dual graph is G.



The “branches” of G have no moduli. As the four nodes or marked points on the component corresponding to the 4-valent vertex move, their cross-ratio sweeps out a $\mathbf{P}^1$ inside $\overline{M}_{0,n}$.



The 4-valent vertex partitions the set $\{1, \ldots, n\}$ into four nonempty subsets:

$$P_C = a \amalg b \amalg c \amalg d$$

Let C_1, C_2 be two F-curves. Then

$$C_1 \sim C_2 \iff P_{C_1} = P_{C_2}.$$

So a partition P specifies a well-defined F-curve class.

Example The class of the F-curve shown above is $[C(\{1, 2, 3\}, \{4, 5\}, \{6, 7\}, \{8\})]$.

When working symmetrically, we often add together all the S_n translates of such a class. Then it is enough record the sizes of the partition of the integer n into four positive parts.

EXAMPLE the symmetrization of $[C(\{1,2,3\},\{4,5\},\{6,7\},\{8\})]$ is $[C(3,2,2,1)]$.

The F-Conjecture

Open Question

What is the nef cone of $\overline{M}_{0,n}$?

The F-conjecture is a conjectural answer to this question.

Conjecture (F-Conjecture)

If D is a divisor on $\overline{M}_{0,n}$, and $D \cdot C \geq 0$ for every F-curve C, then D is nef.

The F-Conjecture is known for n=4,5,6. Also possibly 7.

Some experts think the F-Conjecture is true, and others think it is false.

The F-Conjecture

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The S_n-equivariant F-Conjecture

We can ask a similar question for symmetric divisors:

Conjecture (S_n-equivariant F-Conjecture)

If D is a symmetric divisor on $\overline{M}_{0,n}$, and $D \cdot C \geq 0$ for every F-curve C, then D is nef.

The S_n-equivariant F-Conjecture has been checked for $n \leq 23$.

The S_n-equivariant F-Conjecture also has important consequences for the birational geometry of $\overline{M}_g$.

QUESTION What if we weaken the conditions that define $\overline{M}_{0,n}$ and allow marked points to collide with each other, or to hit the nodes?

Weighted pointed curves

Definition

A *weighted pointed curve* $(C, P_1, \ldots, P_n, A)$ consists of:

- C, a reduced connected projective algebraic curve with at worst nodes as singularities,
- points P_i which lie on C and are ordered; not necessarily smooth/distinct
- $A = (a_1, \ldots, a_n)$, where $a_i \in \mathbb{Q}$ and $0 \leq a_i \leq 1$

A weighted pointed curve is *DM-stable* if the following are satisfied:

- if P_i is a node, then $a_i = 0$,
- if a subset of the points collides, then the sum of their weights is ≤ 1
- the $\mathbb{Q}$-line bundle $\omega(\sum a_i P_i)$ is ample on C.
  - This implies that if $g = 0$ the sum of the weights must be > 2.

$\overline{M}_{g,A}$

- Isomorphism classes of DM-stable weighted pointed curves form a moduli space $\overline{M}_{g,A}$
- Hassett: $\overline{M}_{g,A}$ is projective
- When $A = (1, \dots, 1)$, $\overline{M}_{g,A} \cong \overline{M}_{g,n}$
- When $A = (0, \dots, 0)$, get fibered power
- If A and B satisfy $b_i \leq a_i$ for all i, there is a birational morphism
$$\overline{M}_{g,A} \to \overline{M}_{g,B}$$

So the spaces $\overline{M}_{g,A}$ are natural and important in studying the birational geometry of $\overline{M}_{g,n}$.

For the rest of the talk, I will focus only on genus 0.

Weights that sum to 2

We saw that Hassett's definition of $\overline{M}_{0,A}$ requires $\sum a_i > 2$.

We can extend this to the case $\sum a_i = 2$ as follows:

Suppose $x_i > 0$ for all i. Then we define

$$\overline{M}_{0,\vec{\mathbf{x}}} := (\mathbf{P}^1)^n /\!/_{\vec{\mathbf{x}}} SL(2)$$

where $(\mathbf{P}^1)^n /\!/_{\vec{\mathbf{x}}} SL(2)$ is the GIT quotient of $(\mathbf{P}^1)^n$ by the group $SL(2)$ with the linearization determined by $\vec{\mathbf{x}}$.

We can think of these spaces $\overline{M}_{0,\vec{\mathbf{x}}}$ as moduli spaces, too.

Each point in the moduli space $\overline{M}_{0,\vec{\mathbf{x}}}$ represents $\mathbf{P}^1$ with n ordered marked points. (That is, there are no nodal curves allowed anymore.)

A set of points $P_{i_1}, \ldots P_{i_k}$ is allowed to collide as long as the weights satisfy

$$x_{i_1} + \cdots + x_{i_k} \leq 1.$$

The main line bundles $L_{\vec{\mathbf{x}}}$

Kapranov: For any $\vec{\mathbf{x}}$ there is a morphism

$$\pi_{\vec{\mathbf{x}}}: \quad \overline{M}_{0,n} \to \overline{M}_{0,\vec{\mathbf{x}}}.$$

(This is part of the proof that $\overline{M}_{0,n}$ is the *Chow quotient* of a Grassmannian.)

By GIT there is a distinguished ample line bundle on $\overline{M}_{0,\vec{\mathbf{x}}}$. Call it $\mathcal{L}_{\vec{\mathbf{x}}}$. Then we can define the main line bundles:

DEFINITION: $L_{\vec{\mathbf{x}}} := \pi_{\vec{\mathbf{x}}}^* \mathcal{L}_{\vec{\mathbf{x}}}$.

The line bundles $L_{\vec{\mathbf{x}}}$ are nef, since they are pullbacks of ample bundles along morphisms. If $x_i > 0$ for all i, they are also big.

Notice: if $a_i \geq x_i$ for all i, we could also pull back along morphisms

$$\pi_{A,\vec{\mathbf{x}}}: \overline{M}_{0,A} \to \overline{M}_{0,\vec{\mathbf{x}}}$$

But we are just pulling everything all the way back to $\overline{M}_{0,n}$.

DEFINITIONS: A divisor D or line bundle L on a projective scheme X is nef if $D \cdot C \geq 0$ for every curve $C \subset X$.

A divisor (or line bundle) L is big if

$$\max_{m \geq 0 : H^0(X, L^m) \neq 0} \{\dim \phi_{L^m} X\} = \dim X$$

EXAMPLE How to extend this to the case when some $x_i = 0$:

Suppose

$$\vec{\mathbf{x}} = (x_1, \ldots, x_{n-1}, 0)$$

where $x_i > 0$ if $i < n$. Let

$$\hat{\mathbf{x}} := (x_1, \ldots, x_{n-1})$$

Let $\pi_n : \overline{M}_{0,n} \to \overline{M}_{0,n-1}$ be the map which forgets the last marked point. Then

$$L_{\vec{\mathbf{x}}} := \pi_n^* L_{\hat{\mathbf{x}}}.$$

Such $L_{\vec{\mathbf{x}}}$ are nef but not big.

UPSHOT We have a large collection of nef line bundles on $\overline{M}_{0,n}$. We don't have to do any new work to prove that they are nef; the nefness follows from hard work done by Mumford and Kapranov long ago.

Example Let

$$\begin{aligned} \alpha &= (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{7}{12}, \frac{7}{12}, \frac{7}{12}) \\ \beta &= (\frac{1}{4} - 2\epsilon, \epsilon, \epsilon, \frac{7}{12}, \frac{7}{12}, \frac{7}{12}) \end{aligned}$$

Then

$$\overline{M}_{0,\alpha} \cong \overline{M}_{0,\beta}.$$

In both cases, any subset of $\{P_1, P_2, P_3\}$ can collide with each other or with one of the points P_4, P_5, P_6
while the points P_4, P_5, P_6 can't collide with each other

Question When do two sets of weights α and β yield the same moduli space?

The GIT complex

ANSWER The space of weights is divided into a finite number of chambers separated by walls. Alexeev and I call this the GIT complex.

If α, β belong to the same chamber, then $\overline{M}_{0,\alpha} \cong \overline{M}_{0,\beta}$.
If α, β belong to different chambers, then $\overline{M}_{0,\alpha} \ncong \overline{M}_{0,\beta}$.

The walls correspond to equations of the form

$$\sum_{k \in J, \# J \geq 2} x_{i_k} = 1$$

Moreover, if $\vec{\mathbf{x}}$ lies in the interior of a chamber, then $L_{\vec{\mathbf{x}}}$ is a convex combination of the $L_{\vec{\mathbf{y}}}$ where $\vec{\mathbf{y}}$ ranges over the walls of the chamber.

The cone inside $\mathrm{Pic}(\overline{M}_{0,n})$ of line bundles of the form $L_{\vec{\mathbf{x}}}$ is generated by those $L_{\vec{\mathbf{x}}}$ such that $\vec{\mathbf{x}}$ is a 0-cell of the GIT complex.

QUESTION: How can we study these line bundles?
ANSWER: We know their intersection numbers with the F-curves.

Let $P = \{a, b, c, d\}$ be a partition of $\{1, \ldots, n\}$ into four nonempty subsets. Let $[C(a, b, c, d)]$ be the corresponding F-curve class on $\overline{M}_{0,n}$. Let $\vec{\mathbf{x}}$ be a set of weights summing to two.

Lemma (Alexeev, June 2008)

Let $x_a = \sum_{i \in a} x_i$, $x_b = \sum_{j \in b} x_j$, *etc. We abbreviate*

$$\min := \min\{x_a, x_b, x_c, x_d\}, \qquad \max := \max\{x_a, x_b, x_c, x_d\}.$$

Then

$$L_{\vec{\mathbf{x}}} \cdot [C(a, b, c, d)] = \begin{cases} 0 & \text{if } \max \geq 1, \\ 2(1 - \max) & \text{if } \max \leq 1 \text{ and } \max + \min \geq 1, \\ 2\min & \text{if } \max \leq 1 \text{ and } \max + \min \leq 1. \end{cases}$$

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EXAMPLE: $n = 6$, $\vec{\mathbf{x}} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Then

$$L_{\vec{\mathbf{x}}} \cdot C(\{1,2,3\},\{4\},\{5\},\{6\}) = 0$$

since

$$x_1 + x_2 + x_3 = 1.$$

We also have

$$L_{\vec{\mathbf{x}}} \cdot C(\{1,2\},\{3,4\},\{5\},\{6\}) = \frac{2}{3}$$

since $\min = \frac{1}{3}$, $\max = \frac{2}{3}$.

Proof of the main intersection formula

Let $\Delta_a : \mathbf{P}^1 \longrightarrow (\mathbf{P}^1)^{\#a}$ be the diagonal map.

Define $\Delta_b, \Delta_c, \Delta_d$ similarly.

$\Rightarrow \Delta = \Delta_a \times \Delta_b \times \Delta_c \times \Delta_d : (\mathbf{P}^1)^4 \longrightarrow (\mathbf{P}^1)^n$

The pullback of the linearization $\vec{\mathbf{x}}$ on $(\mathbf{P}^1)^n$ is $\vec{\mathbf{x}}' = (x_a, x_b, x_c, x_d)$.

Thus, the pullback of $\mathcal{L}_{\vec{\mathbf{x}}}$ is $\mathcal{L}_{\vec{\mathbf{x}}'}$.

We get a commutative diagram:

$$\begin{tikzcd}
\overline{M}_{0,n} \arrow[r, "\pi_{\vec{\mathbf{x}}}"] & (\mathbf{P}^1)^n /\!/_{\vec{\mathbf{x}}} \mathrm{SL}(2) \\
\overline{M}_{0,4} \arrow[u, "i"] \arrow[r, "\cong"] & (\mathbf{P}^1)^4 /\!/_{\vec{\mathbf{x}}'} \mathrm{SL}(2) \arrow[u]
\end{tikzcd}$$

The right arrow comes from the map Δ.

The bottom arrow is an isomorphism because both varieties are isomorphic to $\mathbf{P}^1$.

The left arrow comes from the fact that $\overline{M}_{0,n}$ is a Chow quotient.

The class of $i_*[\overline{M}_{0,4}]$ in $\overline{M}_{0,n}$ is $[C(a,b,c,d)]$.

We need to compute the degree of the sheaf $\mathcal{L}_{\vec{\mathbf{x}}'}$ on $\mathbf{P}^1$.

Use the Gelfand-MacPherson correspondence

$$(\mathbf{P}^1)^4 /\!/_{\vec{\mathbf{x}}'} \mathrm{SL}(2) \cong Gr(2,4) /\!/_{\vec{\mathbf{x}}'} T$$

where $T = \mathbb{G}_m^4 / \mathrm{diag}\, \mathbb{G}_m$.

The Plücker embedding

$$Gr(2,4) \hookrightarrow \mathbf{P}^5$$

descends to a map

$$Gr(2,4) /\!/_{\vec{\mathbf{x}}'} T \hookrightarrow \mathbf{P}^5 /\!/_{\vec{\mathbf{x}}'} T$$

The GIT quotient $\mathbf{P}^5 /\!/_{\vec{\mathbf{x}}'} T$ is the polarized toric variety corresponding to the fiber over the point $\vec{\mathbf{x}}' = (x_a, x_b, x_c, x_d)$ of the polytopal map from the simplex $\sigma_6 \subset \mathbb{R}^6$ with 6 vertices to the hypersimplex $\Delta(2,4) \subset \mathbb{R}^4$ (an octahedron).

To finish the proof:

One can compute the fibers of this map explicitly.

Each fiber is either empty, or else it is a dilated triangle (e.g. $\mathbf{P}^2$). One can compute the factor of dilation, and the formula follows.

QUESTION: What are the line bundles $L_{\vec{\mathbf{x}}}$ useful for?

Simpson's results

The Log Minimal Model Program in algebraic geometry begins with a pair (X, D) and produces a sequence of birational maps from X to "simpler" spaces.

Theorem (Simpson, May 2008)

Assume the S_n-equivariant F-conjecture. Then the spaces $\overline{M}_{0,A}$ with $A = (a, a, \ldots, a)$ are log canonical models for $(\overline{M}_{0,n}, \Delta)$.

REMARK: Fedorchuk and Smyth have given a proof of this result (October 2008) which does not use the S_n-equivariant F-conjecture.

SIMPSON'S PROOF: Need to

1. Compute $A_\alpha := \pi_A^*(K_{\overline{M}_{0,A}} + \alpha E)$.
   Show that $K_{\overline{M}_{0,n}} + \alpha\Delta - A_\alpha$ is effective and π_A-exceptional.
2. Show A_α is ample on $\overline{M}_{0,A}$.

Simpson proves 2. by showing that

- A_α is nonnegative on F-curves
- A_α only contracts π_A-exceptional curves.

QUESTION (Alexeev): Are these A_α are in the cone generated by the GIT line bundles?

If so, then this would give an easy proof that A_α is nef on $\overline{M}_{0,n}$, and remove one of Simpson's uses of the S_n-equivariant F-conjecture.

Introducing symmetry

So far, our GIT line bundles $L_{\vec{\mathbf{x}}}$ have no S_n symmetry.

Since our target divisors A_α are symmetric, it is natural to work with symmetrizations of the the $L_{\vec{\mathbf{x}}}$.

DEFINITION: $S_{\vec{\mathbf{x}}} := \bigotimes_{\sigma \in S_n} L_{\sigma\vec{\mathbf{x}}}$.

One can compute intersection numbers for $S_{\vec{\mathbf{x}}}$ by computing the intersection numbers for every constituent $L_{\sigma\vec{\mathbf{x}}}$ and adding them up.

PROBLEM: there are so many permutations!

Special case: if the set of weights has the form:

$$\begin{gathered}(n-1) \text{ entries equal to } a \\ 1 \text{ entry equal to } b \\ (n-1)a + b = 2\end{gathered}$$

then

$$S_{\vec{\mathbf{x}}} = (n-1)!(\otimes_{i=1}^{n} L_i)$$

where L_i is $L_{\vec{\mathbf{x}}}$ where $\vec{\mathbf{x}}$ is the n-tuple of this form with the b in the i^{th} position.
We gave these a special notation:

Definition: $V(a,n) := \bigotimes_{i=1}^{n} L_i$.

Example: $V(1/4,7)$ is

$$L_{(\frac{2}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})} \otimes L_{(\frac{1}{4},\frac{2}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})} \otimes \cdots \otimes L_{(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{2}{4})}.$$

$V(a,n)$ makes sense for any rational a in the appropriate range, but we have had good success in just using the following values.

Let $f = \lfloor \frac{n}{2} \rfloor$.

The dimension of the vector space of divisor classes of symmetric divisors on $\overline{M}_{0,n}$ is $f-1$.

If n is odd, then we consider the $f-1$ line bundles

$$V\left(\frac{1}{f+1}, n\right), V\left(\frac{1}{f+2}, n\right), \ldots, V\left(\frac{1}{n-2}, n\right).$$

If n is even, then we consider the $f-1$ line bundles

$$V\left(\frac{1}{f}, n\right), V\left(\frac{1}{f+1}, n\right), \ldots, V\left(\frac{1}{n-2}, n\right).$$

Example: $n = 9$

Here $f = \lfloor n/2 \rfloor = 4$.

There are three line bundles to consider:
$V(1/5, 9)$, $V(1/6, 9)$, $V(1/7, 9)$.

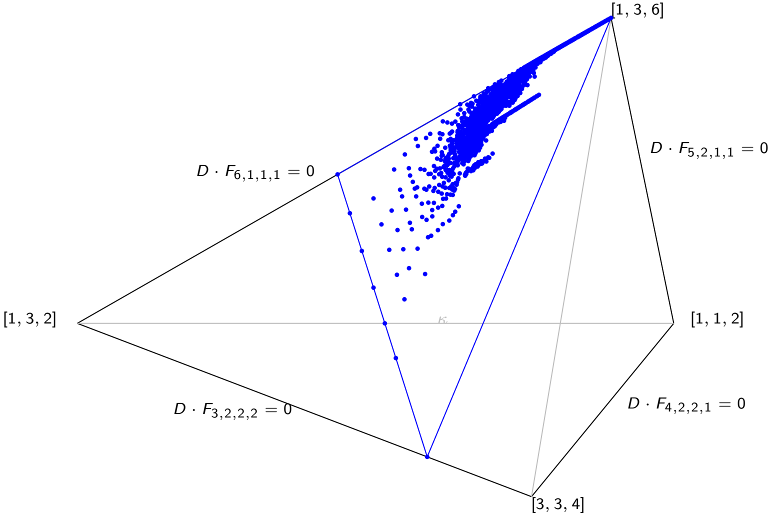
We intersect these with three F-curves $C(4, 3, 1, 1)$, $C(5, 2, 1, 1)$, and $C(6, 1, 1, 1)$ to get the matrix of intersection numbers:

$$\begin{array}{cccc} & V(1/5,9) & V(1/6,9) & V(1/7,9) \\ C(4,3,1,1) & 2 & 2/3 & 4/7 \\ C(5,2,1,1) & 0 & 4/3 & 4/7 \\ C(6,1,1,1) & 0 & 0 & 6/7 \end{array}$$

Results

- For $n = 5$, the GIT cone is the nef cone of $\overline{M}_{0,5}$, and $V(1/2, 5)$ spans the symmetric nef cone.
- For $n = 6$, the GIT cone is strictly smaller than the nef cone of $\overline{M}_{0,6}$, and the symmetrized GIT cone is strictly smaller than the symmetric nef cone.
- For $n = 7, 8, 9$, the symmetrized GIT cone is strictly smaller than the symmetric nef cone of $\overline{M}_{0,n}$.

Symmetrized GIT cone, n=9



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- For $n = 6$, the GIT cone is strictly smaller than the nef cone of $\overline{M}_{0,6}$, and the symmetrized GIT cone is strictly smaller than the symmetric nef cone.
- For $n = 7, 8, 9$, the symmetrized GIT cone is strictly smaller than the symmetric nef cone of $\overline{M}_{0,n}$.
- However, lots of interesting divisors, including Simpson's A_α, lie in the GIT cone.

Proof that the A_α are in the symmetrized GIT cone:

Straightforward calculation.

Consider the F-curve classes

$$[C_i] := [C(n-i-2, i, 1, 1)], \quad i = 1, \dots, f-1.$$

These form a basis of $H_2(\overline{M}_{0,n}, \mathbb{Z})^{S_n}$. We computed the matrix of intersection numbers $U = C_i \cdot V(a, n)$. It is full rank (and upper triangular). We computed the vector of intersection numbers $C_i \cdot A_\alpha$, and solved the system of equations

$$U \cdot (z_1, z_2, \dots, z_k)^T = C_i \cdot A_\alpha.$$

We can write the solution $(z_1, z_2, \dots, z_k)$ as a product of nonnegative numbers. Hence A_α is in the cone spanned by the $V(a, n)$'s.

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Consider the F-curve classes

$$[C_i] := [C(n-i-2, i, 1, 1)], \quad i = 1, \ldots, f-1.$$

These form a basis of $H_2(\overline{M}_{0,n}, \mathbb{Z})^{S_n}$. We computed the matrix of intersection numbers $U = C_i \cdot V(a,n)$. It is full rank (and upper triangular). We computed the vector of intersection numbers $C_i \cdot A_\alpha$, and solved the system of equations

$$U \cdot (z_1, z_2, \ldots, z_k)^T = C_i \cdot A_\alpha.$$

We can write the solution $(z_1, z_2, \ldots, z_k)$ as a product of nonnegative numbers. Hence A_α is in the cone spanned by the $V(a,n)$'s.

We can get a little more from these calculations:

We found a closed formula for the inverse of the matrix of intersection numbers $U = C_i \cdot V(a,n)$. Thus, given a symmetric divisor D, we can easily test whether D is in the cone spanned by the $V(a,n)$'s:

1. Compute the vector of intersection numbers $(C_i \cdot D)$.
2. Multiply $U^{-1}(C_i \cdot D)$.
3. If the resulting numbers are all nonnegative, then D is in the cone generated by the $V(a,n)$'s, hence inside the symmetrized GIT cone.

Conjecture

The symmetrized GIT cone is spanned by the $f-1$ line bundles

$$V\left(\frac{1}{f+1}, n\right), V\left(\frac{1}{f+2}, n\right), \ldots, V\left(\frac{1}{n-2}, n\right),$$

if n is odd, or

$$V\left(\frac{1}{f}, n\right), V\left(\frac{1}{f+1}, n\right), \ldots, V\left(\frac{1}{n-2}, n\right),$$

if n is even.

The conjecture is known for $n = 5, 6, 7, 8, 9$.

If true in general, the conjecture tells us that the symmetrized GIT cone is simplicial.

Also, it is easy to test if a symmetric divisor is in the cone generated by these $f-1$ $V(a, n)$'s. So, if the conjecture is true, then we have an easy test for whether a divisor is in the symmetrized GIT cone.

Open Question

What maps appear in the log minimal model program for $(\overline{M}_{0,n}, D)$ *where* D *is is an effective but non-symmetric sum of boundary components?*

It seems that our methods might be suited to answer this question. Indeed, we naturally began with unsymmetric line bundles $L_{\vec{\mathbf{x}}}$ and then *added* symmetry.

A natural place to start might be $S_a \times S_b$-symmetric divisors D.

December 2008: Alexeev–Swinarski posted to arXiv

January 2008: Alexeev visited Tata Institute, Mumbai, India.

OBSERVATION (Najmuddin Fakhruddin): “For $n = 6$, there are 127 GIT line bundles $L_{\vec{\mathbf{x}}}$ (up to isomorphism).”

“I have a construction which produces 128 line bundles: the 127 GIT line bundles, plus one more.”

Fakhruddin's construction comes from conformal field theory.

To a triple $(\mathfrak{g}, \ell, \overline{\lambda})$, where

- $\mathfrak{g}$ is a simple Lie algebra
- $\ell \in \mathbb{N}$ is called the level
- $\overline{\lambda}$ is an *n*-tuple of dominant integral weights $\overline{\lambda}$ in the Weyl alcove P_ℓ of $\mathfrak{g}$,

one can associate a vector bundle $\mathbb{V}(\mathfrak{g}, \ell, \overline{\lambda})$ on $\overline{M}_{g,n}$.

The fiber of $\mathbb{V}(\mathfrak{g},\ell,\overline{\lambda})$ over $(C,p_1,\dots,p_n)$

Let $\widehat{\mathfrak{g}}$ be the affine Lie algebra associated to $\mathfrak{g}$:

$$\widehat{\mathfrak{g}} = (\ \mathfrak{g}\otimes\mathbb{C}((z_i))) \oplus \mathbb{C}c$$

To each weight λ_i we may associate an irreducible $\widehat{\mathfrak{g}}$-module $\mathcal{H}_{\lambda_i}$. Let $\mathcal{H}_{\overline{\lambda}} = \otimes_{i=1}^{n}\mathcal{H}_{\lambda_i}$.

Let $U = C - \{p_1,\dots,p_n\}$. Choose a local coordinate z_i near each p_i; this gives ring homomorphisms $\mathcal{O}(U)\to\mathbb{C}((z_i))$.

Ueno defines a specific $\mathfrak{g}\otimes\mathcal{O}(U)$ action on $\mathcal{H}_{\overline{\lambda}}$, and

the fiber of $\mathbb{V}(\mathfrak{g},\ell,\overline{\lambda})$ *over a point* $(C,p_1,\dots,p_n)$
is the vector space of coinvariants
of the $\mathfrak{g}\otimes\mathcal{O}(U)$ *action on* $\mathcal{H}_{\overline{\lambda}}$.

Proposition

1. *The vector space of coinvariants is finite dimensional.*
2. *These fibers fit together to form an algebraic vector bundle on* $M_{g,n}$*.*
3. *This vector bundle extends across the boundary to give an algebraic vector bundle on* $\overline{M}_{g,n}$*.*
4. *The fiber over a nodal curve admits a very specific direct sum decomposition known as the factorization rules.*
5. *The vector bundle* $\mathbb{V}(\mathfrak{g}, \ell, \overline{\lambda})$ *on* $\overline{M}_{g,n}$ *admits a projectively flat connection.*
6. *When* $g = 0$*, the connection satisfies the KZ equations.*
7. *When* $g = 0$*, the vector bundle* $\mathbb{V}(\mathfrak{g}, \ell, \overline{\lambda})$ *is globally generated.*

For $g = 0$ Fakhruddin computed Chern class formulas, and formulas for intersection numbers with F-curves.

A taste of Fakhruddin's formulas

$$\mathbb{D}(\mathfrak{g},\ell,\overline{\lambda})\cdot C(a,b,c,d) = \sum_{\overline{\mu}\in P_\ell^4}\deg(\mathbb{V}_{\overline{\mu}})r(\lambda_a\mu_1^*)r(\lambda_b\mu_2^*)r(\lambda_c\mu_3^*)r(\lambda_d\mu_4^*)$$

$$\deg(\mathbb{V}_{\overline{\lambda}}) = \frac{1}{2(\ell+h^\vee)}\Bigg(r_{\overline{\lambda}}\sum_{i=1}^{4}c(\lambda_i) - \sum_{\lambda\in P_\ell}c(\lambda)\left(r(\lambda_1\lambda_2\lambda)r(\lambda_3\lambda_4\lambda^*)+r(\lambda_1\lambda_3\lambda)r(\lambda_2\lambda_4\lambda^*)+r(\lambda_1\lambda_4\lambda)r(\lambda_2\lambda_3\lambda^*)\right)\Bigg)$$

$h^\vee$ = dual Coxeter number of $\mathfrak{g}$
P_ℓ = Weyl alcove in $\mathfrak{g}$ of level ℓ
$c(\lambda)$ = Casimir constant associated to λ
λ^* = weight that gives dual representation to λ
$r(\overline{\lambda})$ = rank of $\mathbb{V}(\mathfrak{g},\ell,\overline{\lambda})$

GIT divisors are conformal blocks divisors

Proposition (Fakhruddin)

The GIT line bundle $L_{\vec{x}}$ is a (multiple of) a conformal blocks determinant for $\mathfrak{g} = \mathfrak{sl}_2$, and any ℓ and $\overline{\lambda}$ such that $x_i = \lambda_i/(\ell + 1)$.

Observation Conformal blocks bundles often give extremal rays of the nef cone and symmetric nef cone.

Open question Where does the cone spanned by conformal blocks determinants lie in the nef cone, and in the symmetric nef cone?

Progress We know that for $n = 6, 7, 8$, the symmetric nef cone is generated by conformal blocks bundles.

Notation: $[2,1]$ below represents the ray spanned by $2B_2 + B_3$.

	Ray	Generator
n=6	$[2,1]$	$\mathbb{D}(\mathfrak{sl}_2, 1, 1^6)$
	$[1,3]$	$\mathbb{D}(\mathfrak{sl}_2, 2, 1^6)$ or $V(1/3, 6)$
n=7	$[1,1]$	$\mathbb{D}(\mathfrak{sl}_2, 2, 2^5 1^2)$
	$[1,3]$	$\mathbb{D}(\mathfrak{sl}_2, 3, 1^6 2)$ or $V(1/4, 7)$
n=8	$[3,2,4]$	$\mathbb{D}(\mathfrak{sl}_2, 1, 1^8)$
	$[2,6,5]$	$\mathbb{D}(\mathfrak{sl}_2, 2, 1^8)$
	$[1,3,6]$	$\mathbb{D}(\mathfrak{sl}_2, 3, 1^8)$ or $V(1/4, 8)$
	$[6,11,8]$	$\mathbb{D}(\mathfrak{sl}_3, 1, \omega_2^7 \omega_1)$
n=9	$[3,3,4]$	$\mathbb{D}(\mathfrak{sl}_2, 2, 2^7 1^2)$
	$[1,3,6]$	$\mathbb{D}(\mathfrak{sl}_2, 4, 1^8 2)$ or $V(1/5, 9)$
	$[1,3,2]$	$\mathbb{D}(\mathfrak{sl}_3, 1, \omega_1^9)$
	$[1,1,2]$	???

$\mathfrak{sl}_9$ and $\mathfrak{so}_{2r+2}$ level 1 conformal blocks, n=9