Graph curves

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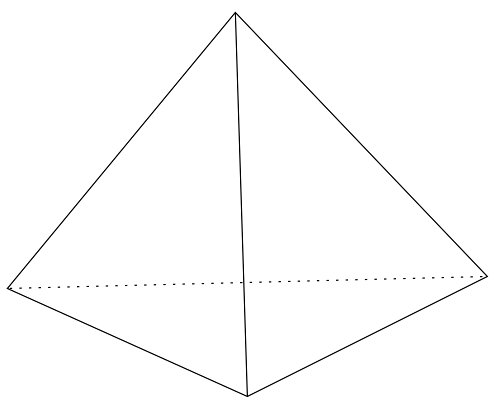
Plan:
I. Define and review graph curves;
present new results;
mention open questions

II. Use in GIT
representations of automorphism groups of curves

Definition 1 *A* graph curve C *is a connected, reduced, projective algebraic curve whose only singularities are nodes, and such that the normalization of every irreducible component is* $\mathbf{P}^1$.

The graph is the *dual graph* of C:

- the irreducible components of C are the vertices of G
- the nodes of C are edges of G



Main reference: Bayer and Eisenbud. "Graph curves." *Adv. Math.* **86** (1991) 1–30.

Running hypothesis: I will assume for the rest of this talk that G has no returning edges (because arithmetic genus 1).

Preliminaries: The genus of a graph curve is the Euler characteristic of the graph.

$$g(C) = \#\text{edges} - \#\text{vertices} + 1$$

In fact, once the edges are oriented, there is an isomorphism

$$\begin{array}{rcl} H^0(C,\omega) & \cong & H^1(G,\mathbb{C}) \\ & & \text{cochain whose value on edge} \\ s & \mapsto & \text{from vertex } \alpha \text{ to vertex } \beta \\ & & \text{representing node } P \text{ is } \mathrm{Res}_{C_\alpha,P}\, s \end{array}$$

[BE] study the multiplication maps

$$\begin{array}{ccccc} H^0(\omega) & \otimes & H^0(\omega) & \to & H^0(\omega^2) \\ & & & & \downarrow \varphi \\ H^1(G,\mathbb{C}) & \otimes & H^1(G,\mathbb{C}) & \to & \mathit{Cochains}^1(G,\mathbb{C}) \end{array}$$

and show that for trivalent graphs, φ is an isomorphim, so

$$H^0(\omega^2) \cong \mathit{Cochains}^1(G,\mathbb{C}).$$

More generally, for trivalent graphs G one has

$$H^0(\omega^n) \cong \mathit{Coch}^1(G,\mathbb{C}) \oplus \bigoplus^{n-2} \mathit{Coch}^0(G,\mathbb{C})$$

Remark: in the stratification of $\partial\overline{\mathcal{M}}_g$ by number of nodes, trivalent graphs correspond to the 0-dimensional strata, i.e. they are as deep in the boundary as possible. Moreover, they do not deform/have no moduli.

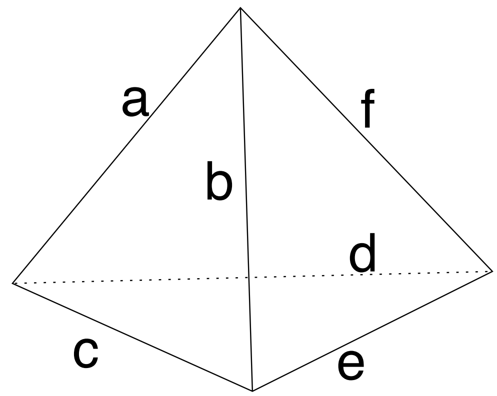
Algorithm for canonical equations of some graph curve:

Let G be trivalent, 2-edge-connected, with no multiple edges (or returning edges). Write
$S := \mathrm{Sym}^\bullet H^0(\omega) \cong \mathrm{Sym}^\bullet H^1(G, \mathbb{C})$
$S \hookrightarrow T := \mathrm{Sym}^\bullet \mathit{Cochains}^1(G, \mathbb{C})$
$R = \bigoplus_n H^0(\omega^n)$
Then $\ker(S \to R) = I(C)$

[BE]: I can be computed as the intersection of S with the ideal J of T generated by all monomials

$$ab, \quad a,b \text{ disjoint edges}$$
$$abc, \quad a,b,c \text{ form a triangle}$$

Example: tetrahedron



Disjoint edge pairs: ae, bd, cf
Triangles: abc, bef, cde, adf.

tetrahedron (continued)

$J = (ae, bd, cf, abc, bef, cde, adf) \subset \mathbb{C}[a..f]$

$S = \mathbb{C}[X, Y, Z]$ where

$$\begin{aligned} X &:= a + b + c \\ Y &:= b + e + f \\ Z &:= c + d + e. \end{aligned}$$

$$\begin{aligned} \Rightarrow I &= \ker(S \to \mathbb{C}[a..f]/J) \\ &= (XYZ(X + Y + Z)). \end{aligned}$$

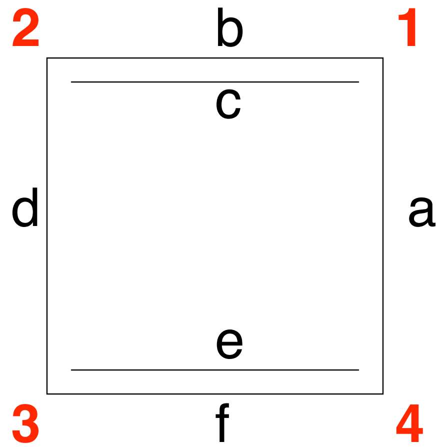
Algorithm [Swinarski]: 2-canonical equations of trivalent graph curves

Label each edge of the graph. If edges a, b, c meet at vertex α, then the ideal $I(C_\alpha)$ of the component C_α represented by α is given by

$$(ab + bc + ac) + (\{\text{edges not meeting vertex } \alpha\}).$$

Then

$$I(C) = \bigcap_{\text{vertices } \alpha} I(C_\alpha)$$



$$I(C_1) = (ab + bc + ac, d, e, f)$$
$$I(C_2) = (bc + cd + bd, a, e, f)$$
$$\vdots$$

Check: genus of C is 3, $\deg \omega^2 = 8$ and Hilbert polynomial of I is $p(t) = 8t - 2$.

4 smooth components, 6 singular points:

{ [1:0:0:0:0:0],. . . [0:0:0:0:0:1] }

Generalize to n-canonical embeddings

• I have an algorithm for $n = 3$, but it's not in final form (the equations aren't obviously symmetric)

• the same strategy works for any $n \geq 2$, with the same beautification problem

Note: in contrast with the [BE] algorithm, the strategy for n-canonical equations works for any trivalent graph. Multiple edges are allowed, and there are no connectivity hypotheses on the graph

Also we can work one component at a time, and it is not necessary to choose a basis of $H^1(G, \mathbb{C})$.

Attempted application in GIT

Bayer–Morrison: Hilbert stability of an ideal I with respect to a 1-PS can be reformulated as whether a certain realization of the state polytope of I contains a specific point.

Thanks to Gfan, it is possible to compute state polytopes in small cases (e.g. 2-canonical embeddings of genus 3 curves). But this only tells us stability with respect to 1-PS inside one maximal torus.

Goal: find a way to conclude stability with respect to the full group $\mathcal{G}$ after a small number of calculations.

Kempf: when y is unstable with respect to $\mathcal{G}$, there is a worst destabilizing 1-PS λ, and λ is preserved by a nontrivial parabolic $\mathcal{P} \subset\neq \mathcal{G}$. So if the stabilizer of y in $\mathcal{G}$ is not contained in any proper parabolic subgroup of $\mathcal{G}$, then y is stable.

Kempf applied this to examples where the representation of the automorphism group is irreducible.

Morrison: if we at least knew that it decomposed into irreducibles all of multiplicity 0 or 1, then we would still "know" $\mathcal{P}$. A little linear programming may allow us to conclude that there is no destabilizing 1-PS.

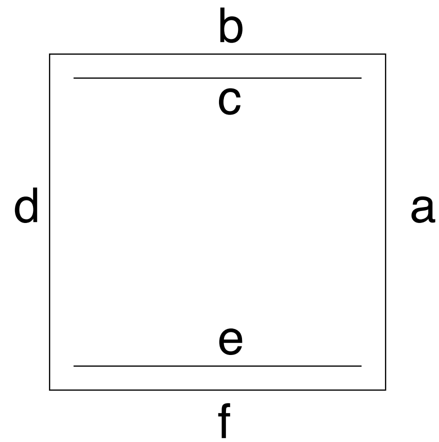
Definition 2 *Say a (connected, reduced, projective) curve C with genus $g > 2$ is a* 0,1-curve for Sym$^2 H^0(\omega^2)$ *if in the decomposition of Sym$^2 H^0(\omega^2)$ into irreducibles with respect to the action of the automorphism group $Aut(C)$ of C, every irreducible has multiplicity* 0 *or* 1.

Problem: Find 0,1-curves.

Theorem 3 (Swinarski) *There are no smooth complex 0,1-curves for Sym*$^2 H^0(\omega^2)$.

But nodal curves can have bigger and more abelian automorphism groups than smooth curves. Maybe we can find examples among trivalent graph curves?

An example will immediately illuminate a necessary condition:



Observe: $a + d$ and $b + c + e + f$ generate two copies of the trivial representation.

Proposition 4 *If there are two disjoint orbits of edges under $Aut(G)$, then $C(G)$ cannot be a 0,1-curve.*

Marston Conder has checked: no trivalent graphs on up to 768 vertices are 0,1-graph curves.

THE END

THANK YOU!

Example: the graph with two vertices, three edges.

$$
\begin{aligned}
I(C_x) &= (y, \\
&\quad ac - ax - 2cx - 2x^2, \\
&\quad bc + bx + 2cx - 2x^2, \\
&\quad ab + ac + bc - ax + bx) \\
I(C_y) &= (x, \\
&\quad ac - ay - 2cy - 2y^2, \\
&\quad bc + by + 2cy - 2y^2, \\
&\quad ab + ac + bc - ay + by) \\
I(C) &= I(C_x) \cap I(C_y)
\end{aligned}
$$