GRÖBNER TECHNIQUES FOR LOW DEGREE HILBERT STABILITY

IAN MORRISON AND DAVID SWINARSKI

1. Introduction

We analyse the Hilbert stability of bicanonical models of certain curves X of small genus with suitably large automorphism groups with respect to linearizations of fixed small degree m. Our examples are very special but they have geometrically interesting applications discussed below.

Our analysis has two main novelties. First, we give a method for deducing the stability, always with respect to $\mathrm{SL}(V)$), of the Hilbert point of a subscheme X of $\mathbb{P}(V)$, from a symbolic calculation of certain state polytopes. Even the possibility of such a reduction for Hilbert points of subschemes of large codimension is new and the hypotheses on $\mathrm{Aut}(X)$ enter into it an essential way.

Second, existing approaches such as, most notably, those pioneered by Gieseker in [11–13] have an asymptotic character and verify Hilbert stability only with respect to linearizations of sufficiently large degree m. Our method allows us to verify stability with respect to linearizations of fixed degree m. The values of m in our main examples are not merely fixed but quite small, typically 6 or less.

The bicanonical curves X and small degrees m in our main examples are chosen because quotients of loci in the bicanonical Hilbert schemes $\mathbb{H}$ in question are predicted to yield new log minimal models of the moduli spaces of stable curves; for further details, see 7.5 of [21]. The results here show the non-emptiness of the stable loci in examples of this type but are far from producing the desired quotients. Therefore, we do not discuss these quotients further here—they will be the subject of a future paper—and deal only with our methods for checking stability and our examples.

Our key assumption on X is that it is 2-*multiplicity free*: the multiplicity, in the natural representation of $\mathrm{Aut}(X)$ on $V = H^0\big(X, K_X^{\otimes 2}\big)$, of every irreducible representation is either 0 or 1. The 2-*multiplicity free* curves that we use as examples are certain special hyperelliptic curves call Wiman curves that are well known in the literature on curves with automorphisms [6], and nodal curves that are joins of 2 or more Wiman curves.

Our approach combines this hypothesis with theorems of Kempf on worst destabilizing 1-ps's to reduce checking stability for the full group $\mathrm{SL}(V)$ to checking stability with respect to a distinguished maximal torus T. There is an easy naive algorithm for checking this symbolically but its complexity makes it impractical except in very simplest cases. By adapting results of Bayer and the first author on state polytopes, we give an algorithm efficient enough that we are able to handle examples arising in our intended applications. The calculations are carried out in `Macaulay2` [27] using the `statePolytope` package of the second author which calls the packages `gfan` [26] and `polymake` [10, 29] to compute intermediate results. Detailed output from our calculations and the source code of our routines are available at the second author's webpage.

Working with small degree m is a sword that cuts both ways. On the one hand, the m we work with are well below the bounds that ensure various standard uniformity hypotheses for ideals of points of $\mathbb{H}$, even those that are deformations of smooth subschemes. A typical example is that the degree m graded pieces of the homogenous ideals do not yield the embedding of $\mathbb{H}$ as a closed subscheme of a Grassmanian that is

Date: Authors' working draft of February 2, 2009.
2000 *Mathematics Subject Classification.* Primary 14L24, 14H10
Secondary 14D22, 13P10.
Key words and phrases. Hilbert stability, state polytope.

used to linearize the relevant PGL-action. Here we have addressed these complications by replacing $\mathbb{H}$ with a multigraded Hilbert scheme $\widehat{\mathbb{H}}$ in the sense of Haiman and Sturmfels [14].

On the other hand, even the algorithm we use is only practical for computing state polytopes in fairly low degrees. It involves computing *all* the monomial initial ideals X (in the coordinates giving the special torus T) and requires a Gröbner basis calculation for *each* initial ideal. In fact, as the genus of X—and hence the bicanonical embedding dimension—increased, we were often unable to carry even these low degree calculations to completion because there are simply too many such ideals. To understand such examples, we use a Monte Carlo strategy that provides an adequate replacement. It computes a random sub-polytope of the state polytope. If X is Hilbert stable and we are fortunate, this sub-polytope provides a proof of stability. This approach can never prove that X is unstable but we are able to do this, when necessary, by educated guesswork. To show that X is unstable, it suffices to find a single destabilizing one-parameter subgroup λ. Geometry—in our examples, analogies with completed stages of the log minimal model program—often suggests what this λ should be. Given the right λ, a *single* Gröbner basis computation suffices to check that λ is destabilizing.

Here is a summary of the plan of the rest of the paper. The details of our multigraded setup, and of the extension to it of the necessary results on state polytopes, are given in section 2. Section 3 reviews Kempf's results on worst one-parameter subgroups and explains how, for multiplicity free X, they reduce checking stability to calculations with state polytopes. The Monte Carlo version of this strategy is described in section 4. Section 5 recalls facts about pluricanonical equations of hyperelliptic curves needed to set up the `Macaulay2` calculations for our Wiman curve examples. These calculations and what they say about stability of the bicanonical models are reviewed in section 6. Finally, we close by listing some ideas for future work.

Acknowledgements. It is a great pleasure to thank Brendan Hassett, David Hyeon, and Yongnam Lee, whose work got us interested in these GIT questions, and Dave Bayer, Johan de Jong, Bill Graham, Anders Jensen, Julius Ross, Greg Smith, Mike Stillman and Bernd Sturmfels for helpful discussions. The first author wishes to acknowledge support from a Fordham University Faculty Fellowship during the early stages of this project. The second thanks Sonja Mapes for her expert instruction on programming in `Macaulay2`, and Dan Grayson for suggesting many improvements to the `statePolytope` package.

2. Hilbert points and state polytopes

Parameter schemes adapted to low degrees. Fix an $(N+1)$-dimensional vector space V and a set of coordinates $\{x_0, \ldots, x_N\}$ identifying V with K^{N+1} and the homogeneous coordinate ring of $\mathbb{P}(V)$ with $S := K[x_0, \ldots, x_N]$. Fix also a Hilbert polynomial P of degree r and let $\mathbb{H}$ be the Hilbert scheme of subschemes $X \subset \mathbb{P}(V)$ with Hilbert polynomial P

The goal of this section is to define state polytopes of such subschemes X—or of their homogeneous ideals $I \subset S$—and to recall their connection to the stability of the Hilbert point of X with respect to the action of $\mathrm{PGL}(V)$ induced by the natural action on $\mathbb{P}(V)$. Both of these notions depend on the choice of the degree m that is used to linearize this action. To make uniform sense of either the Hilbert point or the state polytope for *all* X having a fixed Hilbert polynomial P—that is, over the whole of $\mathbb{H}$—it is necessary to take $m \geq m_P$ for some sufficiently large m_P.

However, the applications we have in mind to stability problems arising in the log minimal model program for $\overline{M}_g$ (cf. [16, 17]) require us to work with a fixed degree $m < m_P$. The main goal of this section is to outline how to transfer the standard constructions to this setting. This is most conveniently achieved by using the multigraded Hilbert schemes constructed by Haiman and Sturmfels [14]. In doing this we have treated general r, since doing so entails no additional complications, but for the applications cited above, we will specialize to the case of curves. We also postpone to a future paper any discussion of what the results here imply about the corresponding GIT quotients of $\mathbb{H}$.

One more remark, before we begin, for those familiar with state polytopes. The variant we define here is based on the construction of Bayer and the first author in [4, §2] which works entirely in a fixed degree. We do not use he state polytopes of Sturmfels [23, Theorem 2.5] which are the Minkowski sum of these for all degrees up to the fixed one.

We begin with a definition of convenience.

Definition 2.1 *An r-dimensional subscheme X of $\mathbb{P}(V)$ with ideal sheaf I is called ℓ-*nice *if*

1. *X is of pure dimension r.*
2. *The natural map $V^\vee \to \Gamma(X, \mathcal{O}_X(1))$ is an isomorphism.*
3. *$\mathcal{O}_X$ is $(\ell-1)$-regular.*
4. *I_X is ℓ-regular.*

The second hypothesis may be viewed more geometrically, as saying that X is embedded in $\mathbb{P}(V)$ by a complete non-degenerate linear series. The third implies, in particular that, for $m \geq \ell$, the sheaf $\mathcal{O}_X(m)$ has no higher cohomology, and hence that its Hilbert polynomial $P(m)$ computes $h^0\big(X, \mathcal{O}_X(m)\big)$. Likewise, the fourth implies that the restriction maps $S_m \to H^0\big(X, \mathcal{O}_X(m)\big)$ are surjective for all $m \geq \ell$ and that I_X is generated be elements of degree at most ℓ.

We denote by $\mathbb{H}_l$ the l-nice locus, in the Hilbert scheme $\mathbb{H}$ of subschemes of $\mathbb{P}(V)$ with Hilbert polynomial P. Fix an l-nice subscheme X. We let $\widehat{R}(m) = \dim_K(S_m) = \binom{m+N}{m}$ and $\widehat{Q}_\ell(m) = \dim_K(I_m)$ for $m \geq \ell$ and $\widehat{Q}_\ell(m) = 0$ for $m < \ell$. In other words, $\widehat{Q}_\ell$ is the Hilbert function of the ideal $\widehat{I}_\ell$ given by truncating I in degrees below ℓ. As usual, we can recover I from any $\widehat{I}_\ell$ by saturating. Our hypotheses imply that $\widehat{I}_\ell$ is generated in degree exactly ℓ. Finally, let $\widehat{P}_\ell(m) = \widehat{R}(m) - \widehat{Q}_\ell(m)$. This is a truncation of the Hilbert function of X and only equals $P(m)$ for $m \geq m_P$.

We denote $\widehat{\mathbb{H}}_\ell$ the multigraded Hilbert scheme of ideals in S with Hilbert function $\widehat{P}_\ell$ and denote by $[I]$ the point of $\widehat{\mathbb{H}}_\ell$ determined by the ideal I. By [14, Corollary 1.2], $\widehat{\mathbb{H}}$ is a projective scheme representing the functor of *locally free* families of such ideals and hence is equipped with a universal family. Their Lemma 4.1 identifies $\widehat{\mathbb{H}}_{m_P}$ with the usual Hilbert scheme $\mathbb{H}$ and, if $\ell < m_P$, then truncation up to degree m_P gives a map $i_\ell : \widehat{\mathbb{H}}_\ell \to \mathbb{H}$.

A few cautions are in order here. First, the l-nice locus in $\widehat{\mathbb{H}}_\ell$ is only locally closed and it need not even be dense—there may be entire components of $\widehat{\mathbb{H}}_\ell$ containing no l-nice ideals.

Second, while $i_\ell(\widehat{\mathbb{H}}_\ell)$ is closed in $\mathbb{H}$ and is injective on the l-nice locus, the map i_ℓ need not be an embedding. This pathology has its origin in the fact that the ideals parameterized by $\widehat{\mathbb{H}}_\ell$ need not be saturated, even in degrees above ℓ where they are not truncated. For example, if $\mathbb{H}$ contains a point X' whose (saturated) ideal I' satisfies $\dim_K(I'_\ell) > \dim_K(I_\ell)$ and $\dim_K(I'_m) = \dim_K(I_m)$ all $m > \ell$, then every choice of a $\dim_K(I_\ell)$-dimensional subspace of I'_ℓ determines an ideal $I'' \in \widehat{\mathbb{H}}_\ell$ mapping to X'. Such examples can be found, for example, with $\mathbb{H}$ the Hilbert scheme of twisted cubics. The upshot is that we cannot replace the ideal I parameterized by a point of $\widehat{\mathbb{H}}_\ell$ by the subscheme X it determines unless we know that the degree ℓ truncation of the saturation of I has Hilbert function exactly $\widehat{P}_\ell$, as we do, by definition, over the l-nice locus.

[When is the monomial initial form $\mathrm{in}_\leq(I)$ of a truncated saturated I determined by a term order $\leq$ also truncated saturated? When it is, then $i_\ell(\mathrm{in}_\leq(I))$ determines the point of $\widehat{\mathbb{H}}_\ell$ parameterizing $\mathrm{in}_\leq(I)$. IM]

The Hilbert matrix. Henceforth we fix values of ℓ and $m \geq \ell$. The first will most commonly be taken to be 2 when we come to our applications. We begin with two remarks designed to lighten our notation. First, since $m \geq \ell$, $\widehat{P}_\ell(m)$ depends only on m so we can and will omit the subscript ℓs used above. Second, we introduce many objects depending on our choice of m in this section, but when there is no risk of confusion, we will omit the m to simplify notation in later sections.

Let $W = \bigwedge^{\widehat{P}(m)} S_m$ and let $\mathbb{G}_m \subset \mathbb{P}(W)$ be the Plücker embedding of the Grassmannian $\mathbb{G}_m := \mathbb{G}\big(\widehat{P}(m), \widehat{R}(m)\big)$ of $\widehat{P}(m)$-dimensional quotient spaces of S_m. There is a map $g_m : \widehat{\mathbb{H}}_\ell \to \mathbb{G}_m$ sending $[I]$ to S_m/I_m. The Plücker map g_m has closed image, but need not be injective. If, however, I is generated in degrees at most m—in particular, for points in the l-nice locus, $g_m([I])$ does determine I.

We want to describe the homogeneous coordinates y_A of $g_m([I]) \in \mathbb{P}(W)$ in a form usable in tools like `Macaulay2`. This is most conveniently and concretly done by working with the *subspace* I_m of S_m rather than the quotient S_m/I_m, and using it to define m-Hilbert matrices $M_{I,m}$. First let $\mathcal{B}_m = \{x^j\}$ be the monomial basis of S_m with a fixed ordering. Then let $\mathcal{C}_m(I) = \{p_i, i = 1, \ldots, \widehat{P}(m)\}$ be any ordered basis of I_m and let $M_{I,m}$ be the $\widehat{P}(m) \times \widehat{R}(m)$ matrix whose ij^{th} entry is the coefficient of the monomial x^j in the equation p_i. The Plücker *coordinates* y_A of are then simply the $\widehat{Q}(m) \times \widehat{Q}(m)$ minors of $M_{I,m}$—one for each Plücker *set* A of $\widehat{Q}(m)$ of the monomials $\mathcal{B}_m$. As in the discussion on page 211 of [4], if $M'_{I,m}$ is the matrix associated to a second basis $\mathcal{C}'_m(I)$ and E is the associated change of basis matrix, then $M' = EM$ and, for all A, $y'_A = \det(E) y_A$. Hence,

(1) The point $g_m([I])$ of $\mathbb{P}(W)$ defined by the collection of y_A is independent of the choice of $\mathcal{C}_m(I)$.
(2) Whether or not any individual y_A vanishes at $g_m([I])$ is likewise independent of this choice.
(3) We may always make this choice so that $M_{I,m}$ is in echelon form.

Example 2.2 For a monomial ideal, we may take the basis $\mathcal{C}_m$ to be monomial, too, and then the Hilbert matrix is particularly simple: it will have exactly one 1 in each row and be 0 otherwise. Thus, for a given m, there is exactly one nonzero Plücker coordinate, given by the Plücker set $A = \mathcal{C}_m$.

Example 2.3 Consider the ideal I of two distinct points in $\mathbb{P}^2$. For instance $P = (1, 2, 3)$ and $Q = (5, 1, -4)$. Let a, b, c be the coordinates on $\mathbb{P}^2$. Then we can view I as $(c - 3a, b - 2a) * (a - 5b, c + 4b)$ and take

$$\mathcal{C}_2 = [2a^2 - 11ab + 5b^2, 8ab - 4b^2 + 2ac, 3a^2 - 15ab - ac + 5bc, 12ab + 3ac - 4bc - c^2]\,.$$

Ordering $\mathcal{B}_{S_2}$ as $[a^2, ab, ac, b^2, bc, c^2]$, we get:

$$M_{I,2} = \begin{pmatrix} 2 & -11 & 0 & 5 & 0 & 0 \\ 0 & 8 & 2 & -4 & -1 & 0 \\ 3 & -15 & -1 & 0 & 5 & 0 \\ 0 & 12 & 3 & 0 & -4 & -1 \end{pmatrix} \tag{1}$$

Then the Plücker point of $M_{I,2}$ is given by the following point, in which we have indexed the Plücker sets by the pair of monomials *omitted* to save space.

$\widehat{12}$	$\widehat{13}$	$\widehat{14}$	$\widehat{15}$	$\widehat{16}$	$\widehat{23}$	$\widehat{24}$	$\widehat{25}$	$\widehat{26}$	$\widehat{34}$	$\widehat{35}$	$\widehat{36}$	$\widehat{45}$	$\widehat{46}$	$\widehat{56}$
45	: -95	: 99	: -154	: 209	: 55	: -18	: 38	: -13	: -83	: 108	: -228	: 22	: 55	: -132

Alternatively, the Plücker coordinates can be computed in `Macaulay2`as follows:

```
i1 : R = QQ[a..c];
i2 : I = intersect(ideal(c-3*a,b-2*a),ideal(a-5*b,c+4*b));
i3 : G = flatten entries super basis(2,I);
i4 : B = flatten entries basis (2,R);
i5 : M = matrix apply(#G, i-> apply(#B, j -> coefficient(B_j,G_i)))
o5 = | 11 -19 9  0   0   0  |
     | 0  1   0  -19 9   0  |
     | 0  0   11 0   -19 9  |
     | 0  0   0  12  -5  -2 |
```

The command `minors(4,M)` then gives the determinants of the 4×4 minors of M. However, the order in which `Macaulay2` lists the basis of $\bigwedge^4 S_2$ is not, in our experience, any obvious one. Note also that, in

`Macaulay2`, we computed the intersection of the two ideals, rather than product. The result is not the ideal we computed by hand, but its saturation. The two Hilbert matrices are different but row-equivalent, and the two sets of Plücker coordinates agree up to a multiple of 121 in each coordinate (that we have supressed in our listing), hence represent the same point in $\mathbb{P}^{14}$.

T-states and state polytopes. We next want to focus on the action of $\mathrm{SL}(V) \cong \mathrm{SL}(N+1)$ on $W := \bigwedge^{P(m)} \mathrm{Sym}^m V$. The Hilbert–Mumford criterion says that $w \in W$ is $\mathrm{SL}(V)$ stable if and only if w is λ-stable for every 1-parameter subgroup (henceforth 1-ps) $\lambda : \mathbb{G}_m \to \mathrm{SL}(V)$. If, in terms of a basis of V with respect to which λ diagonalizes as $\mathrm{diag}(t^{r_0}, \cdots, t^{r_N})$, the point w has coordinates $(w_0, \ldots, w_N)$, we set

$$\mu^L(w, \lambda) := -\min\{r_i \mid i \text{ such that } w_i^* \neq 0\},$$

and w is λ-stable if and only if $\mu^L(w, \lambda) < 0$.

Remark 2.4 A word about the minus in the definition of μ. Our preferred sign convention for the index μ of a Hilbert point w is that of [12], [2] and [21] in which we consider the Grassmanian as parameterizing $P(m)$-dimensional quotients of S_m, given by restriction to $H^0\big(X, \mathcal{O}_X(m)\big)$, and w is stable if any λ acts with negative weight on some non-zero coordinate of w. The minus sign has been inserted to compensate for the fact that here we will be calculating weights of the action of λ on the degree m piece of the ideal I of X which is dimension $\widehat{Q}(m)$. This, of course, gives rise to a quotient of dimension $\widehat{P}(m)$ and the complement of each Plücker set A of monomials gives a basis of this quotient. But, because we are taking m small, we can no longer identify the quotient with $H^0\big(X, \mathcal{O}_X(m)\big)$ except on the l-nice locus and it therefore seemed easier to us to simply work with I_m. This choice has no effect on the notion of $\mathrm{SL}(V)$-stability because the possibility of replacing λ by its inverse means that w is stable if and only, we can always find a non-zero coordinate of w on which λ acts with a weight of either sign.

The connection with Gröbner theory comes via another way of expressing the stability of w with respect to the maximal torus T of $\mathrm{SL}(V)$ determined by a choice of basis B of V. Any character $\chi \in \mathrm{Hom}(T, \mathbb{G}_m)$ of T may be written

$$\chi\big(\mathrm{diag}(d_0, \ldots, d_N)\big) = \prod_{i=0}^{N} d_i^{z_i}.$$

where the z_i are integers, determined (since we are in $\mathrm{SL}(V)$) up to a common shift. Further, any representation W of T decomposes into a direct sum of character eigenspaces W_χ, where $w \in W_\chi$ if and only if $t \cdot w = \chi(t) w$ for all $t \in T$.

Define the *T-state* $\mathrm{State}_T(w)$ of w to be the set of characters for which the eigencomponent w_χ of w is non-zero, and define the *T-state polytope* $\mathcal{P}_T(w)$ to be the convex hull of $\mathrm{State}_T(W)$ in $\mathrm{Hom}(T, \mathbb{G}_m)$.

The group of 1-parameter subgroups of T is dual to its character group: $\lambda \cdot \chi$ is λ-weight of χ—the power of t determined by the homomorphism $\chi \circ \lambda : \mathbb{G}_m \to \mathbb{G}_m$. Viewing λ as giving a linear functional on $\mathrm{Hom}(T, \mathbb{G}_m)$, we may rephrase our discussion of the Numerical Criterion as saying that $w \in W$ is stable with respect to a 1-ps λ in T if and only w has two non-zero eigencomponents w_χ whose character vectors lie on opposite sides of the hyperplane on which λ vanishes. Thus we arrive at the following characterization of GIT stability:

Criterion 2.5 *A vector $w \in W$ is T-stable iff the the trivial character lies in the interior of the state polytope and is T-strictly semistable iff the trivial character lies on the boundary of the state polytope.*

To interpret Criterion 2.5, for Hilbert points, first observe that each eigenspace $(S_m)_\chi$ of S_m is spanned by a single B- monomial M and, if we normalize the choice of the z_i above by requiring that they sum to m then we may identify the character χ and the exponent vector of M. The Plücker coordinates y_A on W likewise give an eigenbasis, although the eigenspaces are no longer 1-dimensional. If we now normalize so that the z_i sum to $\widehat{Q}(m)m$, then we can identify the corresponding character χ_A with the sum of the exponent vectors of the $\widehat{Q}(m)$ monomials determined by y_A. For example, in $\bigwedge^2 \operatorname{Sym}^2 K[a,b,c,d]$, the wedge product $a^2 \wedge bc$ lies in the weight space corresponding to $W_{(2,1,1,0)}$.

Monomials and Plücker coordinates also diagonalize the actions of a 1-ps λ of T on S_m and W. The weight $w_\lambda(M)$ of a monomial M is the sum of the weights of its coordinate factors and the weight $w_\lambda(y_A)$ of a Plücker coordinate y_A is the sum of the weights of the monomials in it. Moreover, these weights agree with the λ-weights of the corresponding characters.

Thus, we think of the characters as lying on the hyperplane

$$Z_m := \{z \in \mathbb{Z}^{N+1} \mid \sum_{i=0}^{N} z_i = mP(m)\}$$

This identifies the trivial character with the point in $\mathbb{Q}^{N+1}$ having all coordinates equal to $\frac{m\widehat{Q}(m)}{N+1}$. In the sequel, we will denote this point by $\mathbf{0}_m$ and call it the *barycenter* of Z_m.

To simplify two notations that we will use frequently, we write $\operatorname{State}_{T,m}(I)$ and $\mathcal{P}_{T,m}(I)$ for the T-state and the T-state polytope of $g_m([I])$, omitting the T when possible.

Criterion 2.6 *The m^{th}-Hilbert point $g_m([I])$ of an ideal I is T-stable iff $\mathbf{0}_m$ lies in the interior of $\mathcal{P}_{T,m}(I)$ and is T-strictly semistable iff $\mathbf{0}_m$ lies on the boundary of $\mathcal{P}_{T,m}(I)$.*

Note that Criteria 2.5 and 2.6 only test T-stability. In Section 3 we will identify a condition under which we can extend this to $\mathrm{SL}(N+1)$-stability.

Example 2.7 The m^{th}-state of a monomial ideal I is a single point, since there is only one nonzero Plücker coordinate. Unless this point equals $\mathbf{0}_m$, the m^{th}-Hilbert point of I is therefore unstable.

Example 2.8 If X a hypersurface of degree d in $\mathbb{P}^N$, we may take $d = m$—so $P(m) = 1$—and suppress the exterior power in W. Both the characters appearing in the decomposition of W and its Plücker coordinates are then indexed by monomials $\prod_{i=0}^{N} x_i^{z_i}$ of degree d, and, viewed as lying in the plane $\sum_{i=0}^{N} z_I = d$, form the d^{th} subdivision of an N-simplex.

Figure 2.9 shows this situation for a cuspidal plane cubic X with equation $x^2z = y^3$. which is unstable with respect to the 1-ps ρ shown. The set of characters appearing in the decomposition is are indicated by dots and the simplex that is their convex hull is the outlined triangle. The state polytope is the line segment joining the two monomials with non-zero coefficients in the equation.

These coordinates are both of which have weight 3 with respect to the 1-ps ρ given by $\rho(t) = \operatorname{diag}(t^4, t, t^{-5})$ and hence this Hilbert point is unstable. The instability is reflected in the fact that $\mathcal{P}(X)$ does not contain $\mathbf{0}$. That $\mathcal{P}(X)$ is degenerate is unimportant: we could add an x^3 term to the equation, making the state polytope the upper sub-triangle subtended by $\mathcal{P}(X)$ without affecting the instability.

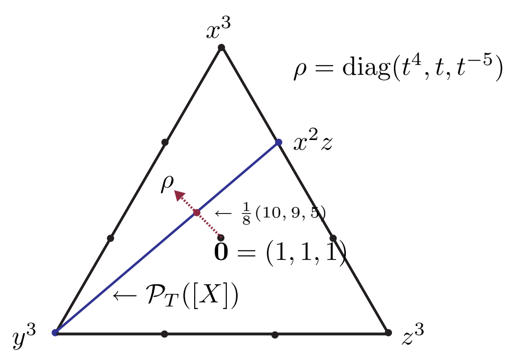

Figure 2.9: Degree 3 state polytope for a cuspical plane cubic with equation $x^2z = y^3$

EXAMPLE 2.10 We continue with Example 2 of the previous section, the ideal of $[1:2:3] \cup [5:1:-4] \subset \mathbb{P}^2$. The nonzero Plücker coordinates hit every possible character, and the state polytope is pictured in Figure 2.11. Here the barycenter,indicated by the central solid circle has coordinates $(\frac{8}{3}, \frac{8}{3}, \frac{8}{3})$.

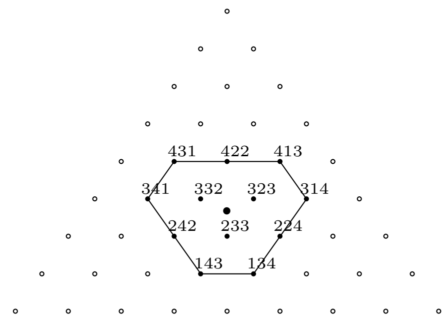

FIGURE 2.11: State$_2(I)$ for two general points in $\mathbb{P}^2$

Vertices of state polytopes and initial ideals. The number of Plücker coordinates grows quickly as the number of variables, the number of generators of the ideal, and m grow. Thus it is impractical to compute the state polytope naively for all but the very simplest examples. The following results, modelled closely on analogous statements in [4], allow us to handle larger examples by giving a procedure for finding the vertices of $\mathcal{P}_T(I_m)$ that avoids the need to deal with interior Plücker coordinates.

Any 1-ps λ in T yields a non-strict order $\geq_\lambda$ on monomials:

$$M \geq_\lambda M' \iff w_\lambda(M) \geq w_\lambda(M').$$

Since the weights w_i of λ on V are integers there will always be ties in large degree. But in any fixed degree m, $\geq_\lambda$ will give a total order for all λ not lying on a finite collection of hyperplanes. If so, we say that λ is m-generic. We will say that λ is generic if it is m-generic for $\ell \leq m \leq m_P$.

Lemma 2.12 *For any I in $\widehat{\mathbb{H}}$ and any generic 1-ps λ, there is a unique Plücker set A_λ of $\widehat{Q}(m)$ monomials such that:*

(1) *y_{A_λ} is non-zero at $g_m([I])$.*
(2) *If $y_{A'}$ is any other Plücker coordinate non-zero at $g_m([I])$, then $w_\lambda(y_A) > w_\lambda(y_{A'})$.*

Moreover, if $M_{I,m}$ is a m-Hilbert matrix for I in echelon form, then the monomials in A_λ span the $>_\lambda$-initial ideal $\in_{>_\lambda}(I)$ of I in degree m.

Proof. This is the content of Lemma 3.3 and Corollary 3.4.(ii) of [4] and the proofs given there apply verbatim in our situation. □

Definition 2.13 *For any generic 1-ps λ, we let $\chi_\lambda = \chi_{A_\lambda}$. In other words, χ_λ is the character given by summing the exponent vectors of the $\widehat{Q}(m)$ monomials in $\mathrm{in}_{>_\lambda}(I)_m$. By (1) of Lemma 2.12, this character is an element of $\mathrm{State}_m(I)$.*

Theorem 2.14 *For any m-generic 1-ps λ, the character χ_λ is a vertex of the state polytope $\mathcal{P}_T(I_m)$. Conversely, if χ is any vertex of $\mathcal{P}_T(I_m)$, then the eigenspace W_χ is one dimensional and is spanned by the Plücker coordinate y_{A_λ} for some m-generic λ. In particular, $\chi = \chi_\lambda$.*

Proof. The inequality in (2) of Lemma 2.12 shows that $\sum_{i=0}^N w_i z_i = w_\lambda(y_A)$ is a supporting hyperplane (χ_λ lies on it and all other χ' in $\mathrm{State}_m(I)$ lie on the negative side of it) and hence proves the first claim Conversely, any supporting hyperplane $\sum_{i=0}^N w_i z_i = b$ to χ may be perturbed so that the coefficients of its normal are the set of weights w_i of a generic 1-ps λ. But then any Plücker coordinate y_A lying in the

χ-eigenspace satisfies the conditions defining y_{A_λ} in Lemma 2.12. The Lemma therefore implies that there is a unique such Plücker coordinate and that $\chi = \chi_\lambda$. The second claim follows. □

Theorem 2.14 is a weaker version of Theorem 3.1 of [4] which shows that if $m \geq m_P$, then the set for vertices of $\mathcal{P}_T(I_m)$ is canonically bijective to the set of initial ideals of I. For the small degrees that we are treating here where the map g_m from an ideal to its degree m graded piece is not injective, a surjection from initial ideals to vertices is all that we can hope for—and all we need to our applications.

By [3, Proposition 1.8], given any multiplicative total order $>$ we can find a 1-ps λ such that $>$ and $>_\lambda$ agree up to degree m. Hence,

Corollary 2.15 *The state polytope $\mathcal{P}_T(I_m)$ is the convex hull of the set of χ_A as A runs over all Plücker sets that are bases for the degree m graded piece of the some initial ideal of I.*

[Dave If I remember several conversations in the fall, you must have an example somewhere in your calculations of two/many different ideals giving the same vertex? Can you extract one? Just a case where there are more initial ideals that vertices would be fine. IM]

Conveniently, Anders Jensen's program `gfan` [26] computes the set of initial ideals of I. Thus, if we compute the m^{th} state of each initial ideal for sufficiently large m, we will have the state polytope. This is what the `Macaulay2` [27] package `StatePolytope` does.

We will not use the following geometric characterization of A_λ but have found it helpful in thinking about the preceding results. The action of $\mathrm{SL}(V)$ on V induces actions on the homogeneous polynomials of each degree on V and hence on Hilbert scheme $\widehat{\mathbb{H}}$ and on the Grassmanian $\mathbb{G}_m$ for which the map g_m is equivariant. Since $\widehat{\mathbb{H}}$ is projective, we can define an ideal J giving a point of $\widehat{\mathbb{H}}$ by

$$[J] := \lim_{t \to 0} \lambda(t) \cdot [I] \, .$$

Lemma 2.12 says that y_{A_λ} is the unique Plücker coordinate that is non-zero at $g_m([J])$ and hence that $g_m([J]) = g_M([\mathrm{in}_{>_\lambda}(I)])$. But all these arguments apply equally to any other degree between ℓ and m_P so that, in all these degrees, J and $\mathrm{in}_{>_\lambda}(I)$ are equal. Hence,

Proposition 2.16 *$J = \mathrm{in}_{>_\lambda}(I)$ in degrees above ℓ.*

3. Kempf's theory of the worst 1-ps

Let x be a point of a scheme X acted on by G. Already in the first edition of [9], Mumford realized that if x is unstable, there is a worst destabilizing 1-ps λ. This λ is not unique; there is a parabolic subgroup P_x for which $\nu^L(x, \lambda) = \nu^L(x, p^{-1}\lambda p)$, where $\nu^L(x, \lambda) = \mu^L(x, \lambda)/||\lambda||^2$.

Kempf and Rousseau proved this picture. Most important for us will be the following results, which are paraphrased from [20, Theorem 3.4 and §5]:

Theorem 3.1 *Suppose V is a representation of a reductive group G, and $x \in \mathbb{P}(V)^{un}$.*

1. *There exists at least one worst destabilizing 1-ps, and a nontrivial parabolic $P_x \subset\neq G$ such that $P_x = P(\lambda)$ for any worst destabilizing 1-ps λ.*
2. *P_x contains $\mathrm{Stab}_x(G)$.*

Remark. We will be applying this result to $V = \bigwedge^{P(m)} \mathrm{Sym}^m K^{N+1}$ and $G = \mathrm{SL}(N+1)$.

Kempf applies these results to conclude stability of Chow and Hilbert points of abelian varieties and homogeneous spaces ([20, Cor. 5.2 and 5.3)]: the representations of the automorphism groups of these varieties are irreducible, so the stabilizer is not contained in any nontrivial parabolic, so these must be GIT stable.

There are very few examples of pluricanonically embedded smooth curves with an automorphism group acting via an irreducible representation. (For instance, a full list of canonical curves with this property is found in [6, App. B]; the highest genus example is $g = 14$. Examples should only get rarer as ν increases.) So, Kempf's strategy must be modified if it is to be applied to moduli of curves.

Here is the most general statement we expect to be true:

Conjecture 3.2 *Suppose the representation of* $\mathrm{Aut}(X) \to \mathrm{SL}(N+1)$ *is multiplicity-free. That is, in the decomposition of this representation into irreducibles, no irreducible has multiplicity greater than 1. Choose a basis* $v_0, \dots, v_N$ *of* K^{N+1} *adapted to the decomposition into irreducibles. Suppose* x *is unstable, so that* P_x *exists. Then the diagonal maximal torus for this coordinate system contains a worst destabilizing* 1*-ps.*

We would apply this conjecture in the following way: Suppose we have an X such that $\mathrm{Aut}(X)$ acts in a multiplicity-free way, and we compute the state polytope and find that it contains the barycenter. Then this establishes the G-stability of X, in addition to T-stability.

We can prove the conjecture in a special case, which, in light of the results of sections to follow, is already enough to give interesting examples.

Proposition 3.3 *Suppose* $\mathrm{Stab}_x(G)$ *contains a finite cyclic subgroup* C *such that the representation* $\rho : C \hookrightarrow \mathrm{SL}(N+1)$ *is multiplicity-free. Suppose the* m^{th} *Hilbert point of* X *is* $\mathrm{SL}(N+1)$*-unstable. Then in any coordinate system diagonalizing the* C *action, the diagonal maximal torus* T_C *contains a worst destabilizing* 1*-ps.*

Proof. The hypotheses of the proposition and Kempf's results (see 3.1 above) ensure that there exist worst destabilizing 1-ps's, that there is a parabolic P_x such that $P_x = P_\lambda$ for any worst destabilizing 1-ps λ, and P_x contains $\mathrm{Stab}_x G$.

Let M be a generator of $\rho(C)$. Since C is cyclic, multiplicity-freeness of this representation is equivalent to M having distinct eigenvalues.

Let λ be a worst destabilizing 1-ps, and let T_λ be a maximal torus diagonalizing λ. Since λ and M might not commute with each other, the two tori T_C and T_λ might not be the same. We consider a change-of-basis matrix Ψ from T_λ to T_C. If $\Psi \in P_x$, then by [9] Proposition 2.7 we know $\mu^L(x, \Psi^{-1}\lambda\Psi) = \mu^L(x, \lambda)$, and thus $\Psi^{-1}\lambda\Psi$ is a worst destabilizing 1-ps in T_C, as desired.

Proposition 3.4 below yields that $\Psi \in P_\lambda$. □

Proposition 3.4 *Let* $P \subset\neq \mathrm{SL}(n)$ *be a parabolic, and suppose* $M \in P$ *has distinct eigenvalues. Then* M *can be diagonalized by a matrix in* P*.*

Proof. In suitable coordinates, P can be put into block form, so that

$$M = \begin{pmatrix} M_1 & * & * & \cdots & * \\ 0 & M_2 & * & \cdots & * \\ 0 & 0 & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & M_k \end{pmatrix},$$

where each M_i is square of shape $n_i \times n_i$, and $n_1 + n_2 + \cdots + n_k = n$.

The characteristic polynomial of M is the product of the characteristic polynomials of the M_i. (The determinant of a block matrix like this is the product of the determinants, and $M - \lambda I$ is again a block matrix.)

It follows that each M_i has distinct eigenvalues, and hence has a full set of eigenvectors.

First, let $\lambda_1, \dots, \lambda_{n_1}$ be the eigenvalues of M_1, and choose a set of corresponding eigenvectors $t_1, \dots, t_{n_1}$. (In particular, each t_i is a vector with n_1 coordinates.) The t_i can be lifted to eigenvectors $v_i, (i = 1, \dots, n_1)$ of M by appending $n - n_1$ zeroes at the end of each of them.

Let $\mu_1, \dots, \mu_{n_2}$ be the eigenvalues of M_2, and choose a set of corresponding eigenvectors $w_1, \dots, w_{n_2}$ (which thus have n_2 coordinates each). We argue that these can also be lifted to eigenvectors $v_j, (j = 1, \dots, n_2)$ of

M. Write

$$v_j = (v_j^{(1)}, \ldots, v_j^{(n_1)}, w_j^{(1)}, \ldots, w_j^{(n_2)}, 0, \ldots, 0)$$

We must solve for the $v_j^{(k)}$. But these are the solutions of the system

$$(M_1 - \mu_j \mathrm{Id}_{n_1}) \cdot \begin{pmatrix} v_j^{(1)} \\ \vdots \\ v_j^{(n_1)} \end{pmatrix} = \begin{pmatrix} -m_{1,n_1+1} w_j^{(1)} - \cdots - m_{1,n_1+n_2} w_j^{(n_2)} \\ \vdots \\ -m_{n_1,n_1+n_2} w_j^{(1)} - \cdots - m_{n_1,n_1+n_2} w_j^{(n_2)} \end{pmatrix}, \tag{2}$$

which exist because the matrix $(M_1 - \mu_j \mathrm{Id}_{n_1})$ has full rank. (μ_j is an eigenvalue of M_2, so by the assumption that the eigenvalues are distinct, μ_j is not an eigenvalue of M_1.)

In a similar way, we can lift eigenvectors for each block M_i to eigenvectors of M. Then the matrix Ψ whose columns are the $\vec{v}_{i,j}$ diagonalizes M and lies in P. □

We believe that with minor modifications this proof can be extended to the case where $\mathrm{Aut}(X)$ contains a finite abelian subgroup whose representation is multiplicity-free (instead of having image a finite cyclic subgroup, as printed above). Indeed, we hope that proving Conjecture 3.2 will not be too much harder.

Remarks.

1. In past conversations, we told many people that we were looking for multiplicity-free representations of $\mathrm{Aut}(X)$ on $\bigwedge^{P(m)} \mathrm{Sym}^m K^{N+1}$, but we later realized that this is not the correct thing to do. The parabolic P_x lives in $G = \mathrm{SL}(N+1)$, not $\mathrm{SL}(\bigwedge^{P}(m) \mathrm{Sym}^m K^{N+1})$.
2. Note that a general curve is as far from being multiplicity-free as possible—a general smooth curve of genus g has a trivial automorphism group, so for any embedding, the corresponding representation is $N+1$ copies of the trivial representation of the trivial group. So this strategy can potentially only prove directly the stability of some special curves. But by the openness of GIT stability and the coarseness of the Zariski topology, proving that a single smoothable curve is stable proves that a general curve is stable!

4. Strategy: Monte Carlo

We apply all these results to compute/prove GIT stability of certain bicanonical curves of small genus for small m linearizations.

In theory, we would find a multiplicity-free ideal I, then compute the state polytope, and check whether it spans the barycenter. But in practice, already for $g = 4$ this plan is too difficult to carry out; there are too many initial ideals.

The remedy is to realize that to check GIT stability, we don't need to know the entire state polytope. If we find enough points in the state of I to span the barycenter, then we are done!

Strategy 1a. If I is suspected to be GIT stable, then compute random Plücker coordinates until their characters span the barycenter.

Problem: The nonzero Plücker coordinates may be sparse. For instance, for the twisted cubic and $m = 3$, one is working in $\mathbb{G}(10, 20)$. There are 184756 Plücker coordinates, and only 324 of these are nonzero.

This problem led us to Corollary 2.15 and the following strategy.

Strategy 2a. If I is suspected to be GIT stable, then compute random initial ideals until the collection of their m^{th} states span the barycenter.

Then, by Corollary 2.15, the state of I spans the barycenter, and I is GIT stable.

On the other hand, to prove GIT instability, one would need to compute the entire state polytope to know that the state does not span the barycenter. This is computationally impractical. But it is generally pretty easy to show something is unstable.

Strategy 2b. If one suspects that I is GIT unstable, then one can probably think of a candidate destabilizing 1-ps λ. Then confirm that λ is destabilizing by computing $\mu^m(I, \lambda)$.

5. Equations of Hyperelliptic Curves

Let C be a smooth hyperelliptic curve. It is well-known that the canonical divisor is ample but not very ample; the corresponding morphism is a 2:1 map onto a rational normal curve. On the other hand, degree considerations show that νK is very ample for any integer $\nu \geq 2$. Let $\phi_{\nu K} : C \to \mathbb{P}^N$ be the corresponding map. Thus, $N+1 = (2\nu-1)(g-1)-1$. What are equations for $\phi_{\nu K}(C)$?

Stevens ([22], page 137–138) and Eisenbud ([8]) both give equations for hyperelliptic curves under more general linear systems than just $|\nu K|$. There is nothing new in this section; the only purpose is to show how quick and easy this is for pluricanonical linear systems.

Let $\pi : C \to \mathbb{P}^1$ be the g^1_2 on C. For convenience, write $k := \nu(g-1)$ and $e := g+1$. Then $\phi_{\nu K}(C)$ lies on the scroll $S = \mathbb{P}_{\mathbb{P}^1}(\pi_*(\nu K))$, where $\pi_*(\nu K) \cong \mathcal{O}(k) \oplus \mathcal{O}(k+e)$; thus $S \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-e))$.

Let C be given by the equation $y^2 = f(x)$, where $f(x)$ is a degree $2g+2$ or $2g+1$ polynomial in x. Then a basis of $H^0(C, \nu K)$ is given by:

$$\begin{gathered} 1, x, x^2, \ldots, x^k \\ y, yx, yx^2, \ldots, yx^{k-e} \end{gathered}$$

We check: $(k+1)+(k-e+1) = 2\nu(g-1)-(g+1)+2 = (2\nu-1)(g-1) = h^0(C, \nu k)$. Abusing notation, we use these basis elements as variables on $\mathbb{P}^N$.

Equations for the scroll are classical; modern references are [15] Exercise 9.11 and [1, pp. 96–100]. Suppose $(\nu-1)(g-1) > 2$, so that $k > e$. Then the scroll equations are given by the 2×2-minors of the *deleted catalecticant matrix*

$$M := \begin{pmatrix} yx^{k-e} & yx^{k-e-1} & \cdots & yx & | & x^k & x^{k-1} & \cdots & x \\ yx^{k-e-1} & yx^{k-e-2} & \cdots & y & | & x^{k-1} & x^{k-2} & \cdots & 1 \end{pmatrix}. \tag{3}$$

Write I_S for the ideal generated by the 2×2-minors of M.

Next, choose a set Q of quadrics encoding the following equations:

$$\begin{aligned} y^2 &= f(x) \\ y^2 x &= x f(x) \\ &\vdots \\ y^2 x^{2(k-e)} &= x^{2(k-e)} f(x). \end{aligned} \tag{4}$$

The particular choices of quadrics used to encode these equations won't matter once these are combined with the scroll equations. Write I_Q for the ideal generated by these quadrics.

Claim 5.1 *Suppose* $\deg f(x) = 2g+1$*, i.e. the hyperelliptic curve is branched at* ∞*. Then* $I(\phi_{\nu K}(C)) = I_S + I_Q$*. That is, the ideal of* $\phi_{\nu K}(C)$ *is given by the scroll equations (the* 2×2*-minors of* M*) together with* $2(k-e)+1$ *additional equations coming from* Q*.*

Example. We find the equations of $\phi_{2K}(C)$, where C is the genus 4 hyperelliptic curve given by $y^2 = x^9 - 1$. (This is the Wiman curve of genus 4, which will be discussed in the next section.)

Here $k = 6$, $e = 5$, and $N = 8$. We coordinatize $\mathbb{P}^8$ as follows:

$$\begin{matrix} 1 & : & x & : & x^2 & : & x^3 & : & x^4 & : & x^5 & : & x^6 & : & yx & : & yx^2 \\ a & : & b & : & c & : & d & : & e & : & f & : & g & : & h & : & i \end{matrix}$$

The deleted catalecticant matrix is

$$\begin{pmatrix} yx & | & x^6 & x^5 & x^4 & x^3 & x^2 & x \\ y & | & x^5 & x^4 & x^3 & x^2 & x & 1 \end{pmatrix} = \begin{pmatrix} i & | & g & f & e & d & c & b \\ h & | & f & e & d & c & b & a \end{pmatrix},$$

yielding

$$\begin{gathered} I_S = (-gh+fi, -fh+ei, -f^2+eg, -eh+di, -ef+dg, -e^2+df, -dh+ci, \\ -df+cg, -de+cf, -d^2+ce, -ch+bi, -cf+bg, -ce+bf, -cd+be, -c^2+bd, \\ -bh+ai, -bf+ag, -be+af, -bd+ae, -bc+ad, -b^2+ac) \end{gathered}$$

(It's easy to do this in `Macaulay2` [27]: Enter M using
`matrix {{i,g,f,e,d,c,b},{h,f,e,d,c,b,a}}`. Then `minors(2,M)` gives I_S.)

We may encode the equation $y^2 = x^9 - 1$ as $h^2 - dg + a^2$, the equation $y^2x = x^{10} - x$ as $hi - eg + ab$, and the equation $y^2x^2 = x^{11} - x^2$ as $i^2 - fg + ac$. Then the ideal we seek is

$$(-gh + fi, -fh + ei, -f^2 + eg, -eh + di, -ef + dg, -e^2 + df, -dh + ci, -df + cg, -de + cf,$$
$$-d^2 + ce, -ch + bi, -cf + bg, -ce + bf, -cd + be, -c^2 + bd, -bh + ai, -bf + ag, -be + af,$$
$$-bd + ae, -bc + ad, -b^2 + ac, h^2 - dg + a^2, hi - eg + ab, i^2 - fg + ac).$$

This ends our discussion of the genus 4 Wiman curve.

In working with these ideals, one notices the following phenomenon.

Claim 5.2 *Order the elements in Q in any order. $V(I_S + (q_0))$ has dimension 1. Each time an additional quadric q_i is added, the degree and the arithmetic genus decrease by 1, until one reaches a smooth curve of degree $4g - 4$ and genus g.*

Clearly, $V(I_S + (q_0))$ must contain the hyperelliptic curve plus a number of secant lines, and as we add more generators to the ideal, we are gradually getting rid of the secant lines. But we have not proven this carefully yet, so we leave it as a claim. (This may be partially or fully explained in [8]; we have not read it carefully enough yet.)

Finally, we note the following corollary:

Corollary 5.3 *For these choices of embeddings of C, and for any $\nu_2 \geq \nu_1 \geq 2$, we get that the ${\nu_2}^{\text{th}}$-canonically embedded image of C projects down to the ${\nu_1}^{\text{th}}$-canonically embedded image of C, where the projection is given by "forgetting the extra variables."*

6. Results

We prove that the bicanonical genus 3 Wiman curve W_3 (defined below) is unstable for $m = 2$ and stable for $m \geq 3$ (which matches the predictions based on [19]). We prove that the genus 4 and 5 Wiman curves W_4 and W_5 are stable for all $m \geq 2$ (which matches predictions of Hassett and Hyeon). We also prove that a specific genus 5 nodal curve with a genus two tail is unstable for $m < 6$, semistable for $m = 6$, and stable for $m \geq 7$.

Bicanonical Wiman curves are multiplicity-free. We write W_g for the *Wiman curve of type I in genus g*. Recall that this is the smooth hyperelliptic curve given by the equation $y^2 = x^{2g+1} - 1$. We can think of W_g as a hypersurface in $\mathbb{P}(1, g+1, 1)$ given by $y^2 = x^{2g+1}z - z^{2g+2}$; then we call $[1 : 0 : 0]$ the branch point at infinity. The automorphism group $\mathrm{Aut}(W_g)$ of W_g is cyclic of order $4g+2$. For some primitive $(4g+2)^{\text{nd}}$ root of unity ζ, $\mathrm{Aut}(W_g)$ acts with weight 1 on y and weight 2 on x. We record two important facts about this action:

Proposition 6.1 $\mathrm{Aut}(W_g)$ *fixes the branch point at infinity.*

Proposition 6.2 *The representation of* $\mathrm{Aut}(W_g)$ *on $H^0(W_g, 2K)$ is multiplicity-free.*

Proof. Recall the basis for $H^0(C, 2K)$ specified in (3) is

$$1, x, x^2, \ldots, x^k$$
$$y, yx, yx^2, \ldots, yx^{k-e}$$

Then $\mathrm{Aut}(W_g)$ acts diagonally on this basis with weights

$$0, 2, 4, \ldots, 2k$$
$$1, 3, 5, \ldots, 2(k-e) + 1$$

Thus, this basis is adapted to the decomposition of the representation of $\mathrm{Aut}(W_g)$ into irreducibles, and the representation is multiplicity-free. ∎

Stability of some smooth Wiman curves.

Example: the genus 3 Wiman curve. Genus 3 bicanonical curves are not explicitly covered by Section 5, since $(\nu-1)(g-1)=2$, or equivalently, $k=e$. But we can stretch the algorithm there to cover this case, too: instead of the curve lying on a scroll given by a deleted catalecticant matrix, in genus 3, the curve lies on a cone over the rational normal curve given by a catalecticant matrix. To this we can add a quadric encoding $y^2 = f(x)$, yielding: the ideal for W_3 in $K[a,b,c,d,e,f]$ is $(ac-b^2, ad-bc, ae-bd, bd-c^2, be-cd, ce-d^2, f^2-ab+e^2)$.

In this case it is actually possible to compute the entire state polytope, so it is not necessary to use Monte Carlo methods. `gfan` [26] finds 4615 initial ideals; interestingly, while I is generated by quadrics, some of the initial ideals have much higher regularity—one of the initial ideals has a generator of degree 19. (The Gotzmann number for the Hilbert polynomial $8t-2$ is 26.) We find that for $m=2$, W_3 is unstable, and for integers $3 \leq m \leq 19$, W_3 is stable.

This corroborates predictions based on work by Hyeon and Lee ([19, Proposition 19]. They find that for $g=3$, divisors of slope $\leq 28/3$ contract the hyperelliptic locus. On the other hand, we can compute the polarization on the quotient: it is

$$[6\nu^2 m - 2\nu m - 2\nu + 1]\lambda - [\frac{1}{2}\nu^2 m]\delta$$

which for $\nu = 2$ gives $[20m-3]\lambda - [2m]\delta$, or slope $\frac{20m-3}{2m}$. We solve $\frac{20m-3}{2m} = \frac{28}{3}$ to obtain $m = 9/4$. Thus, one predicts that a hyperelliptic curve is unstable for $m=2$ and stable for $m \geq 3$, which matches what is found.

We can go one step further in this case: for $m=2$ we can find the closest point (or *proximum*) on the state polytope to the barycenter. This can be computed using the Maple package `Convex` ([24]). The result is $\vec{p} = (\frac{12}{5}, \frac{12}{5}, \frac{12}{5}, \frac{12}{5}, \frac{12}{5}, \frac{10}{5})$. The average of these entries is $7/3$ (the barycenter), and $\vec{p} - (7/3, \ldots, 7/3) = (1/15, \ldots, 1/15, -1/3)$. Thus, the worst 1-ps is one which scales the span of the rational normal curve with equal weights and scales the cone with complementary weight.

We check: using the `MUm` function from [18], and $w = (10,10,10,10,10,12)$ we get $\mathtt{MUm}(I,w,2) = -4$ and $\mathtt{MUm}(I,w,3) = 24$. By [16] Prop 3.17, this gives $\mu([W_3]_m, \lambda) = 4(m-1)(4m-9)$. Thus this 1-ps is destabilizing for $m < 9/4$.

What about $m > 19$? Well, by results of Gieseker we know that smooth curves of degree $d > 2g+1$ embedded by complete linear systems are asymptotically GIT stable. Moreover, for a given 1-ps λ, we know that the weight of λ with respect to the linearization given by m is a quadratic polynomial in m, and one of its roots is $m=1$ ([16] Prop. 3.17). Gieseker's theorem tells us the leading term has positive sign. Thus, if our polynomial is positive for some $m_0 > 1$, we know it is positive for all $m \geq m_0$. Thus, for hyperelliptic curves, we can use $m_0 = 3$.

Is this the best we can do? What happens for $9/4 < m < 3$? We predict that the worst 1-ps for $m=2$ is the last one to become stable as m increases, and hence W_3 is stable for $m > 9/4$, but we don't have a clear enough picture for fractional linearizations to prove this yet.

Example: the genus 4 Wiman curve. We computed the ideal of W_4 in Section 5: it is

$$\begin{gathered}(-gh+fi, -fh+ei, -f^2+eg, -eh+di, -ef+dg, -e^2+df, -dh+ci, -df+cg, -de+cf,\\ -d^2+ce, -ch+bi, -cf+bg, -ce+bf, -cd+be, -c^2+bd, -bh+ai, -bf+ag, -be+af,\\ -bd+ae, -bc+ad, -b^2+ac, h^2-dg+a^2, hi-eg+ab, i^2-fg+ac).\end{gathered}$$

Using the Monte Carlo strategy in `Macaulay2` [27], we find that W_4 is stable for $m = 2,3,4,5,6,7$. As discussed above, these calculations, combined with the arguments of [12, 16] imply that W_4 is stable for all $m \geq 2$.

For $g \geq 4$ Hyeon predicts (private communication) that divisors of slope ≤ 9 contract the hyperelliptic locus. Solving $\frac{20m-3}{2m} = 9$ yields $m = 3/2$, so the prediction is that this curve should be stable for all $m \geq 2$, which matches what is found.

Example: the genus 5 Wiman curve. W_5 behaves just like for W_4. The ideal is obtained using the methods of Section 5. We find that W_5 is stable for $m = 2, 3, 4, 5, 6, 7$. Arguing as above, this implies that W_5 is stable for all $m \geq 2$, which matches as predicted by Hassett and Hyeon.

Example: A genus 5 curve with a genus 2 tail. *Disclaimer: we assume throughout the discussion of this example that Proposition 3.3 can be extended from cyclic to abelian groups.*

Here we consider an example of a nodal genus 5 curve which has a genus 3 component and a genus 2 component (hence a genus 2 tail). Hassett and Hyeon predict that such a curve is stable for $m > 6$, semistable for $m = 6$, and unstable for $m < 6$. Our findings uphold this prediction.

We used the following nodal curve X: Let W_3 be the Wiman curve of genus 3, and let P be the branch point at infinity. Let W_2 be the genus 2 Wiman curve, and let Q be the branch point at infinity. Then $X = W_3 \cup_{P=Q} W_2$.

ω_X^2 is very ample, and the image of X under the corresponding morphism ϕ is a degree 16 curve in $\mathbb{P}^{11}$. We need to find equations for $\phi(X)$ and show that the action of $\mathrm{Aut}(X)$ is multiplicity-free.

We know that $\omega_X^2|_{W_3} = \omega_{W_3}^2(2P)$, and $\omega_X^2|_{W_2} = \omega_{W_2}^2(2Q)$. In principle, [22] and [8] explain how to write equations for such curves; however, we used `MAGMA` [5, 28].

For W_3, coordinatize $\mathbb{P}^7$ using the variables a–h, and map $W_3 \to \mathbb{P}^7$ by

$$\begin{array}{ccccccccccccccc} yx^6 & & yzx^5 & & z^5x^5 & & z^4x^6 & & z^3x^7 & & z^2x^8 & & zx^9 & & 1x^{10} \\ a & : & b & : & c & : & d & : & e & : & f & : & g & : & h \end{array}$$

Then P maps to $[0:0:0:0:0:0:0:1]$. The ideal of W_3 in $\mathbb{P}^7$ is given by

$$\begin{array}{l} (-g^2 + fh, -fg + eh, -f^2 + eg, -f^2 + dh, -ef + dg, -ef + ch, -e^2 + df, -e^2 + cg, \\ -de + cf, -d^2 + ce, ag - bh, af - bg, ae - bf, ad - be, ac - bd, \\ b^2 + c^2 - fg, ab + cd - g^2, a^2 + d^2 - gh) \end{array}$$

and the generator of $\mathrm{Aut}(W_3)$ acts on a–h with weights $5, 3, 10, 12, 0, 2, 4, 6$. Then the ideal of W_3 in $\mathbb{P}^{11}$ is obtained by adding (i, j, k, l) to the ideal above, and the $\mathrm{Aut}(W_3)$-action is extended to $\mathrm{Span}\{i, j, k, l\}$ by giving these weight 6.

For W_2, coordinatize $\mathbb{P}^4$ using the variables h–l, and map $W_2 \to \mathbb{P}^4$ by

$$\begin{array}{ccccccccc} 1x^4 & & x^3z & & x^2z^2 & & xz^3 & & yx \\ h & : & i & : & j & : & k & : & l \end{array}$$

Then Q maps to $[1:0:0:0:0]$. The ideal of W_2 in $\mathbb{P}^4$ is given by

$$(l^2 - hi + k^2, i^2 - hj, ij - hk, j^2 - ik),$$

and the generator of $\mathrm{Aut}(W_2)$ acts on h–l with weights $8, 6, 4, 2, 7$. Then the ideal of W_2 in $\mathbb{P}^{11}$ is obtained by adding (a, b, c, d, e, f, g) to this ideal, and the $\mathrm{Aut}(W_2)$-action is extended to $\mathrm{Span}\{a, b, c, d, e, f, g\}$ by giving these weight 8.

Putting this together, the ideal of X is

$$\begin{array}{rcl} I(X) & = & (gl, fl, el, dl, cl, bl, al, gk, fk, ek, dk, ck, bk, \\ & & ak, gj, fj, ej, dj, cj, bj, aj, gi, fi, ei, di, ci, bi, ai, \\ & & g^2 - fh, fg - eh, eg - dh, dg - ch, ag - bh, f^2 - dh, ef - ch, df - cg, \\ & & af - bg, e^2 - cg, de - cf, ae - bf, d^2 - ce, ad - be, ac - bd, \\ & & b^2 + c^2 - eh, ab + cd - fh, a^2 + ce - gh, \\ & & j^2 - ik, ij - hk, i^2 - hj, hi - k^2 - l^2) \end{array}$$

We check that the $\mathrm{Aut}(X)$-action is multiplicity-free. Let ζ_{10} and ζ_{14} be primitive 10^{th} and 14^{th} roots of unity, respectively. Let $K = \mathbb{Q}[\zeta_{10}, \zeta_{14}]$. Then the $\mathrm{Aut}(X)$ representation is generated by the diagonal matrices

$$D(\zeta_{14}^5, \zeta_{14}^3, \zeta_{14}^{10}, \zeta_{14}^{12}, \zeta_{14}^0, \zeta_{14}^2, \zeta_{14}^4, \zeta_{14}^6, \zeta_{14}^6, \zeta_{14}^6, \zeta_{14}^6, \zeta_{14}^6)$$

and

$$D(\zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^8, \zeta_{10}^6, \zeta_{10}^4, \zeta_{10}^2, \zeta_{10}^7).$$

One can check in `MAGMA` [28] or `GAP` [25] that this representation is indeed multiplicity-free. Here are commands for doing this in `MAGMA` [28]:

```
K<z>:=CyclotomicField(140);
u:=z^14;
t:=z^10;
GL12K:=GeneralLinearGroup(12,K);
D1:=DiagonalMatrix([t^5,t^3,t^10,t^12,t^0,t^2,t^4,t^6,t^6,t^6,t^6,t^6]);
D2:=DiagonalMatrix([u^8,u^8,u^8,u^8,u^8,u^8,u^8,u^8,u^6,u^4,u^2,u^7]);
G:=sub<GL12K | D1,D2>;
Gmod:=GModule(G);
chi:=Character(Gmod);
X:=CharacterTable(G);
Decomposition(X,chi);
```

Note that Proposition 3.3 does not cover this example, since $\mathrm{Aut}(X)$ is not cyclic. However, we assume that Proposition 3.3 extends to finite abelian groups and forge ahead.

We ran our Monte Carlo stability program for small values of m. For $m < 6$, we expect that X is unstable (see next paragraph), and indeed, the program timed out before finding initial ideals that spanned the barycenter. For $m = 6$, we found that X is semistable. For $m = 7$, X must be stable (once again this follows from the $m = 6$ calculation), but our program timed out before it found enough initial ideals to span the barycenter.

Next, we proved by hand that X is unstable for $m < 6$. Let λ be the 1-ps which acts with weights $w = 6, 6, 6, 6, 6, 6, 6, 6, 4, 2, 0, 5$. This gives average weight $\alpha = 59/12$. The Hilbert polynomial is $P(m) = 16m - 4$. The weight filtration on $\mathcal{O}(m)$ looks like

$$6m, 6m, ...6m, 6m - 1, 6m - 2, 6m - 3, ..., 6, 5, 4, 2, 0,$$

giving $w(m) = 78m^2 - 15m - 4$. Putting this all together gives

$$w(m) - mP(m)\alpha = (-2/3)(m - 1)(m - 6)$$

with the desired roots and sign for asymptotic stability with a flip at $m = 6$.

Alternatively, using the `MUm` function from [18] yields:

$$\begin{aligned}
\mathtt{MUm}(I, w, 2) &= -32\\
\mathtt{MUm}(I, w, 3) &= -48\\
\mathtt{MUm}(I, w, 4) &= -48\\
\mathtt{MUm}(I, w, 5) &= -32\\
\mathtt{MUm}(I, w, 6) &= 0\\
\mathtt{MUm}(I, w, 7) &= 48
\end{aligned}$$

and then their formula (2.6), gives $\mu([C]_m, \rho) = 8(m - 1)(m - 6)$.

Note that the two calculations differ by a constant -12; we have not completely reconciled this yet, but under each set of authors' sign convention, the respective calculation shows that X is unstable for $m < 6$.

7. What next?

(1) Clearly, the next step should be to prove stability for an arithmetic genus 5 curve with a ramphoid cusp for $m < 6$, and semistability for $m = 6$. We tried an irreducible example as well as a genus three curve glued to a rational component with a ramphoid cusp. The Monte-Carlo program timed out in both cases before finding enough initial ideals to span the barycenter. We hope that if we analyze the output for the genus 5 curve with a genus 2 tail more carefully, perhaps we will be able to identify some key initial ideals, and then use these to help prove stability for the ramphoidal examples.

(2) More generally, we hope that we might be able to go from random calculations to deterministic proofs. For instance, for the Wiman curves, can we identify geometrically a small number of initial ideals which span the barycenter? Then perhaps this geometrically meaningful cluster will span the barycenter for any g, giving us a proof in higher genus.

(3) Thanks to openness of semistability and the coarseness of the Zariski topology, the calculations presented above tell us a lot more than just stability of Wiman curves:

Proposition 7.1 *A general smoothable curve of Hilbert polynomial $P(t) = 8t - 2$, $P(t) = 12t - 3$, or $P(t) = 16 - 4$ is stable for $m \geq 3$.*

We note that the curves need not be canonically embedded, nor smooth. (Must the linear system be complete?) We hope that techniques like those found in [7] could be used to construct moduli spaces.

(4) It would be nice to prove the Bayer–Morrison theorem for small m, or find a counterexample, and hence streamline Section 2.

(5) It would be nice to find more multiplicity-free curves than just bicanonical Wiman curves. Are there multiplicity-free nonhyperelliptic canonical curves? Are there any GIT stable multiplicity-free ribbons?

We know one place not to look: Bicanonical trivalent graph curves (the "vital points" of $\overline{M}_g$) are never multiplicity-free for $g < 101$. We can write equations for them, and it is easy to see that a multiplicity-free trivalent graph curve would have to be vertex and edge-transitive. Marston Conder did a search for us, and there are no such graphs, at least with $g < 101$.

(6) It would be interesting to know how the theory behaves in positive characteristic. Computing over a finite field ought to be faster than computing over $\mathbb{Q}$. Curves in positive characteristic can have automorphism groups which exceed the Hurwitz bound; perhaps this could make it easier to find multiplicity-free examples. On the other hand, representation theory in characteristic p may be harder than in characteristic 0. And is knowing GIT stability in char p for a small number of p's enough to conclude it in char 0?

References

[1] Enrico Arbarello, Maurizio Cornalba, Phillip Griffiths, and Joe Harris, *Geometry of algebraic curves. Vol. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 267, Springer-Verlag, New York, 1985. MR770932 (86h:14019) ←11

[2] Elizabeth Baldwin and David Swinarski, *A geometric invariant theory construction of moduli spaces of stable maps*, Int. Math. Res. Pap. IMRP **1** (2008), Art. ID rpn 004, 104, available at arXiv:0706.1381v3. MR2431236 ←5

[3] Dave Bayer, *The division algorithm and the Hilbert scheme*, Harvard University, 1982, http://www.math.columbia.edu/~bayer/thesis/thesis.html. ←8

[4] David Bayer and Ian Morrison, *Standard bases and geometric invariant theory. I. Initial ideals and state polytopes*, J. Symbolic Comput. **6** (1988), no. 2-3, 209–217. Computational aspects of commutative algebra. MR988413 (90e:13001) ←3, 4, 7, 8

[5] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265. Computational algebra and number theory (London, 1993). MR1484478 ←14

[6] Thomas Breuer, *Characters and automorphism groups of compact Riemann surfaces*, London Mathematical Society Lecture Note Series, vol. 280, Cambridge University Press, Cambridge, 2000. MR1796706 (2002i:14034) ←1, 9

[7] Lucia Caporaso, *A compactification of the universal Picard variety over the moduli space of stable curves*, J. Amer. Math. Soc. **7** (1994), no. 3, 589–660. MR1254134 (95d:14014) ←16

[8] David Eisenbud, *Transcanonical embeddings of hyperelliptic curves*, J. Pure Appl. Algebra **19** (1980), 77–83. MR593247 (82f:14027) ←11, 12, 14

[9] John Fogarty, Frances Kirwan, and David Mumford, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR1304906 (95m:14012) ←8, 9

[10] Ewgenij Gawrilow and Michael Joswig, *polymake: a framework for analyzing convex polytopes*, Polytopes—combinatorics and computation (Oberwolfach, 1997), DMV Sem., vol. 29, Birkhäuser, Basel, 2000, pp. 43–73. MR1785292 (2001f:52033) ←1

[11] David Gieseker, *Global moduli for surfaces of general type*, Invent. Math. **43** (1977), no. 3, 233–282. MR0498596 (58 #16687) ←1

[12] David Gieseker, *Lectures on moduli of curves*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 69, Published for the Tata Institute of Fundamental Research, Bombay, 1982. MR691308 (84h:14035) ←1, 5, 13

[13] David Gieseker, *Geometric invariant theory and applications to moduli problems*, Invariant theory, 1983, pp. 45–73. Proceedings of the 1st 1982 Session of the Centro Internazionale Matematico Estivo (CIME) held at Montecatini, June 10–18, 1982. MR718124 (84i:14004) ←1

[14] Mark Haiman and Bernd Sturmfels, *Multigraded Hilbert schemes*, J. Algebraic Geom. **13** (2004), no. 4, 725–769, available at arXiv:math/0201271. MR2073194 (2005d:14006) ←2, 3

[15] Joe Harris, *Algebraic geometry: a first course*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1995. Corrected reprint of the 1992 original. MR1416564 (97e:14001) ←11

[16] Brendan Hassett and Donghoon Hyeon, *Log minimal model program for the moduli space of stable curves: the first flip* (2008), available at arXiv:0806.3444v1. ←2, 13

[17] Brendan Hassett and Donghoon Hyeon, *Log canonical models for the moduli space of stable curves: first divisorial contraction* (2006), available at arXiv:math/0607477. To appear in Trans. Amer. Math. Soc. ←2

[18] Brendan Hassett, Donghoon Hyeon, and Yongnam Lee, *Stability computation via Gröbner bases* (2007), available at http://math.rice.edu/∼hassett/papers/GroebnerGIT/stabcomp.pdf. ←13, 15

[19] Donghoon Hyeon and Yongnam Lee, *Log minimal model program for the moduli space of stable curves of genus three* (2007), available at arXiv:math/0703093. ←12, 13

[20] George R. Kempf, *Instability in invariant theory*, Ann. of Math. (2) **108** (1978), no. 2, 299–316. MR506989 (80c:20057) ←8

[21] Ian Morrison, *GIT constructions of moduli spaces of stable curves and maps* (2008), available at arXiv:0810.2340v1. ←1, 5

[22] Jan Stevens, *Deformations of singularities*, Lecture Notes in Mathematics, vol. 1811, Springer-Verlag, Berlin, 2003. MR1976016 (2004b:32049) ←11, 14

[23] Bernd Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996. MR1363949 (97b:13034) ←3

Software Packages Referenced

[24] Matthias Franz, *Convex: a Maple package for convex geometry* (2006), available at http://www.fourier.ujf-grenoble.fr/~franz/convex/. Version 1.1.2. ←13

[25] The GAP Group, GAP*: Groups, Algorithms, and Programming, a system for computational discrete algebra* (2008), available at http://www.gap-system.org. Version 4.4.11. ←15

[26] Anders Jensen, gfan*: a software package for computing Gröbner fans and tropical varieties* (2008), available at http://www.math.tu-berlin.de/~jensen/software/gfan/gfan.html. Version 0.3. ←1, 8, 13

[27] Dan Grayson and Mike Stillman, Macaulay 2*: a software system for research in algebraic geometry* (2008), available at http://www.math.uiuc.edu/Macaulay2/. Version 1.1. ←1, 8, 12, 13

[28] Computational Algebra Research Group, School of Mathematics and Statistics, University of Sydney, MAGMA *computational algebra system* (2008), available at http://magma.maths.usyd.edu.au/magma/. Version 2.15-1. ←14, 15

[29] Ewgenij Gawrilow and Michael Joswig, polymake*: a framework for analyzing convex polytopes* (2007), available at http://www.math.tu-berlin.de/polymake/. Version 2.3. ←1

Department of Mathematics, Fordham University, Bronx, NY 10458
E-mail address: morrison@fordham.edu

Department of Mathematics, University of Georgia, Athens, GA 30602
E-mail address: davids@math.uga.edu