CODE SAMPLES TO ACCOMPANY "GRÖBNER TECHNIQUES FOR LOW DEGREE HILBERT STABILITY"

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This file consists mainly of annotated code samples for performing various calculations discussed in our paper "Gröbner techniques for low degree Hilbert stability" (available at http://www.arxiv.org/abs/0910.2047) and linked to from that paper. These examples are intended to give readers not familiar with some of the packages we use models for calculations needed in our examples. We claim no originality—nor even any cleverness in coding. We have also included a few lengthy ideal descriptions omitted from the paper.

1. Ideals used in our paper

Here we record some ideals used in our calculations, but, to save space, not listed in the paper.

1.1. **The ideal $\mathcal{I}'$ of the elliptic bridge in Example 8.6.** Recall that this ideal is given in coordinates diagonalizing the two generating automorphisms.

$$\begin{aligned}
\mathcal{I}' \;=\; & (DG+GI, CG-GJ, CG+GJ, BG-GK, BG+GK, AG-GL, AG+GL, DF+FI, CF-FJ,\\
& CF+FJ, BF-FK, BF+FK, AF-FL, AF+FL, E^2-EF-2EH-FH+H^2,\\
& E^2-DF-H^2+FI, DE-DH+EI-HI, CE-CH-EJ+HJ, CE-CH+EJ-HJ,\\
& BE-BH-EK+HK, BE-BH+EK-HK, AE-AH-EL+HL, AE-AH+EL-HL,\\
& -DE+E^2+DH+2EH+H^2+EI-HI, DE-F^2-G^2+DH-EI-HI,\\
& DE+DH+EI+HI, CE+CH-EJ-HJ, CE+CH+EJ+HJ, BE+BH-EK-HK,\\
& BE+BH+EK+HK, AE+AH-EL-HL, AE+AH0+EL+HL, BD-BI-DK+IK,\\
& BD-BI+DK-IK, AD-AI-DL+IL, AD-AI+DL-IL,\\
& -CD+D^2+CI+2DI+I^2-DJ+IJ, BD+BI-DK-IK, BD+BI+DK+IK,\\
& AD+AI-DL-IL, AD+AI+DL+IL, C^2-CD+D^2-CI-I^2-2CJ-DJ-IJ+J^2,\\
& BC-BJ-CK+JK, BC-BJ+CK-JK, AC-AJ-CL+JL, AC-AJ+CL-JL,\\
& B^2-BC-BJ-2BK-CK-JK+K^2, B^2-AC-AJ-K^2+CL+JL,\\
& -AB+B^2+AK+2BK+K^2+BL-KL, 2A^2-BC-BJ+CK+JK+2L^2)
\end{aligned}$$

1.2. **The ideal I of the genus 5 nodal curve in with a genus 2 Weierstrass tail in Example 8.7.**

$$\begin{aligned}
I \;=\; & (gl, fl, el, dl, cl, bl, al, gk, fk, ek, dk, ck, bk, ak,\\
& gj, fj, ej, dj, cj, bj, aj, gi, fi, ei, di, ci, bi, ai,\\
& g^2-fh, fg-eh, eg-dh, dg-ch, ag-bh, f^2-dh, ef-ch, df-cg,\\
& af-bg, e^2-cg, de-cf, ae-bf, d^2-ce, ad-be, ac-bd,\\
& b^2+c^2-eh, ab+cd-fh, a^2+ce-gh,\\
& j^2-ik, ij-hk, i^2-hj, hi-k^2-l^2)
\end{aligned}$$

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1.3. **The ideal I of the genus 5 nodal curve with a general genus 2 tail in Example 8.9.**

$$\begin{aligned} I(C) \quad = \quad & (gl, fl, el, dl, cl, bl, al, gk, fk, ek, dk, ck, bk, ak, \\ & gj, fj, ej, dj, cj, bj, aj, gi, fi, ei, di, ci, bi, ai, \\ & g^2 - fh, fg - eh, eg - dh, dg - ch, ag - bh, f^2 - dh, ef - ch, df - cg, \\ & af - bg, e^2 - cg, de - cf, ae - bf, d^2 - ce, ad - be, ac - bd, \\ & b^2 + c^2 - eh, ab + cd - fh, a^2 + ce - gh \\ & k^2 - jl, hk - il, hj - ik, i^2 - jk - 2hl) \end{aligned}$$

2. Computing Plücker coordinates in Macaulay2

To compute the Plücker coordinates of Example 2.4 we proceed as follows.

```
i1 : R = QQ[a..c];
i2 : I = intersect(ideal(c-3*a,b-2*a),ideal(a-5*b,c+4*b));
i3 : G = flatten entries super basis(2,I);
i4 : B = flatten entries basis (2,R);
i5 : M = matrix apply(#G, i-> apply(#B, j -> coefficient(B_j,G_i)))
o5 = | 11 -19 9  0   0   0  |
     | 0  1   0  -19 9   0  |
     | 0  0   11 0   -19 9  |
     | 0  0   0  12  -5  -2 |
i6 : flatten entries gens minors(4,M)
o6 = {15972, -6655, 27588, 1573, -25289, -2662, -13068, -4598,
     18634, 10043, 2178, -11979, -6655, 11495, -5445}
i7 : gcd(oo)
o7 = 121
i8 : o6/o7
o8 = {132, -55, 228, 13, -209, -22, -108, -38, 154, 83, 18, -99, -55, 95, -45}
```

The order in which Macaulay2 lists the basis of $\bigwedge^4 S_2$ does not match the convention used in our paper. Note also that, in Macaulay2, we computed the intersection of the two ideals, rather than the product. The result is not the ideal we computed by hand, but its saturation. The two Hilbert matrices are different but row-equivalent, and the two sets of Plücker coordinates agree up to a multiple of 121 in each coordinate (that we supressed in our listing), hence represent the same point in $\mathbb{P}^{14}$.

3. Using the package StatePolytope in Macaulay2

Here we show how to use the package `StatePolytope` in `Macaulay2`.

```
i1 : loadPackage("StatePolytope");
i2 : R=QQ[a..d];
i3 : I = ideal(a*c-b^2, a*d-b*c, b*d-c^2);
i4 : initialIdeals(I)
o4 = {{b*d, a*d, a*c}, {c^2, a*d, a*c}, {c^2, b*c, a*c, a^2*d}, {c^2,
b*c, b^3, a*c}, {c^2, b*c, b^2}, {b*d, b^2, a*d}, {b*d, b*c, b^2,
a*d^2}, {c^3, b*d, b*c, b^2}}
i5 : statePolytope(2,I)
i6 : statePolytope(3,I)
```

To see $\mathrm{State}_m(I)$ for larger values of m, see Kapranov-Sturmfels-Zelevinsky, page 202. Note that they compute state polytopes using monomials outside the initial ideal rather than inside, but it is simple to convert between the two conventions: simply subtract each vertex from the vector $\frac{((m+N-1)!m}{m!N!}(1,\ldots,1)$.

4. Testing whether N ideals span the barycenter

If a point Q is contained in a polytope $\mathcal{P} \subset \mathbb{R}^n$, then there exists a set of n vertices $\{v_j\}_{j=1}^n$ of $\mathcal{P}$ such that Q is contained in ConvexHull$\{v_j\}$.

In our Monte Carlo calculations, of course, it is more difficult to find exactly n points whose convex hull contains the barycenter, but the calculations required are easier: rather than calling polymake or another convex geometry package, we can test this containment using linear algebra.

Let L be the list of n initial ideals. Below we check whether $\{\mathrm{State}_3(I_j)\}$ contains $\mathbf{0}_3$ in its convex hull.

```
i1 : R=QQ[a..i];
i2 : L = {
{a*h^2, d*f^3, a*f^2, d*g, c*d, a*g, g*h^2, c*h^2, b*h, a*e, h*i, b*f, e*h, f*i,
 c*f, e*f, b^2, b*i, b*g, i^2, b*c, b*e, c*i, c*g, e*i, e*g, c^2, c*e, e^2},
{b^3, e*h, b*h, b^2*c, a*b^2, c*e, a*e, e*g, b*g, f^2, f*h, d*f, c*f, a*f, f*g,
 d*h, d^2, c*h, g*h, c*d, a*d, d*g, c^2, a*c, a^2, c*g, a*g},
{d^4, d^2*g, a*e^2*i, c*d^2, d*g*h, a*g, c*e, c*d*h, e*f, b*e, d*f, d*i, b*d, a*f,
 h^2, c*g, b*g, f*h, h*i, c^2, b*h, c*f, c*i, b*c, f^2, f*i, i^2, b*f, b*i, b^2},
{b^5*i^2, c^5*f, c^3*e^2, b^2*e, c^2*e*f, b*e^2, a*c, b*f, a*i^2, a*e, e^3, c*f^2,
 a*f, e^2*f, b*h, b*d, c*h, b*g, c*d, f^3, c*g, a*d, e^2*h, a*g, d*e, h*i, e*g,
 d*i, f*h, d*f, f*g, h^2, d*h, d^2, g*h, d*g},
{c^4, a*b*f, b^2*f*h, a^2*f, a*c, a*f*h, c*h, a*i, c*e, c*d, b*e, b*d, a*e, a*d,
 c*g, b*g, i^2, a*g, h*i, e*i, h^2, e*h, d*h, e^2, d*e, d^2, g*h, e*g, d*g},
{b*f, b*e, a*d, c^2, c*f, c*e, e^2, d*h^2, d*g, e*h, a*c, b*g, a*f, a*e, b*i,
 c*h^2, f*h^2, c*g, e*g, c*i, e*i, a^2, g*h, h*i, a*g, a*i, i^2},
{d^3, c*d^2, c^2, b*g*h, b*e, d*e, c*e, b*f, d*f, c*f, i^2, e^2, a*g, f*i, e*f,
 a*d, a*c, d*h, c*h, f^2, a*i, a*e, h*i, e*h, a*f, f*h, h^2},
{b*d, c*e, c*f, b^2, a*d, c*g, b*e, b*f, e^2, d*h, b*i, e*f, a*e, d*g^2, b*g, a*f,
 e*g, a*i, i^2, a*g, e*h, f*h, h*i, g*h, h^2},
{b^3*e^3, b^4, c*f*h^2, b^3*c, c^3, e^2*f, d*h^2, c^2*f, a*b^2, e*f^2, c*f^2, b*f,
 a*e, a*c, f^3, d*e, b*d, h*i, c*d, a*f, e*i, e*g, b*i, a^2, c*i, b*g, c*g, d*f,
 a*d, f*i, a*i, a*g, d^2, d*i, d*g, i^2}
};
i3 : m=3;
i4 : Z = apply(#L, i -> subhp(m,ideal(L_j)));
i5 : bary = apply(9,i -> sum(Z_0)/9);
i6 : Z = transpose matrix Z;
i7 : v = (Z^-1) * matrix(transpose {bary});
i8 : apply(9, i -> v_0_i >=0) == apply(9, i -> true)
```

In line `i4`, we create a matrix whose rows are the 3^{rd} Hilbert points of the initial ideals contained in the list `L`. In line `i5`, we create the barycenter $\mathbf{0}_3$. Lines `i6` and `i7` write $\mathbf{0}_3$ in the basis given by the 3^{rd} Hilbert points, and call this answer `v`. Finally, in line `i9`, we check that all coordinates of `v` are nonnegative.

5. Writing equations for curves in MAGMA

Here we demonstrate how to get equations of a hyperelliptic curves under an embedding of the form $|aK + bP|$. Specifically, we take $C = \mathcal{W}_3$ and the linear series $|2K + 2P|$ where P is the point at infinity.

```
L<z>:=CyclotomicField(14);
R<x>:=PolynomialRing(L);
C:=HyperellipticCurve(x^7-1);
K:=CanonicalDivisor(C);
P:=C![1,0,0];
D:=2*K+2*Divisor(P);
phi:=DivisorMap(D);
WP2<x,y,z>:=Domain(phi);
P7<a,b,c,d,e,f,g,h>:=Codomain(phi);
phi;
Ideal(phi(C));
```

Note that in writing the resulting equations in Example 8.7, we have reordered the variables to conveniently attach the genus 2 tail.

6. Equations of scrolls in Macaulay2

In our paper, we see that ideals of bicanonical curves can be written as the ideal of a scroll plus some additional quadrics.

We briefly mention that there is an easy way to get the ideal of such scrolls.

Take for instance the genus 4 Wiman curve $\mathcal{W}_4$. The ideal I_S for the associated scroll can be obtained with the Macaulay2 commands

```
M = matrix {{i,g,f,e,d,c,b},{h,f,e,d,c,b,a}};
IS = minors(2,M)
```

7. Computing the proximum with Convex in Maple

Here is an example of how to find the proximum (closest point on a polytope to a point outside it) using the package `Convex` in `Maple`.

```
with(convex):
P:=convhull([0,0],[1,0],[0,1],[1,1]):
Q:=Vector([2,0]):
proximum(P,Q);
quit:
```

8. Checking multiplicity freeness in MAGMA

Here is a sample code sample showing how to check that a reducible Wiman is multiplicity free in MAGMA
.

```
K<z>:=CyclotomicField(140);
u:=z^14;
t:=z^10;
GL12K:=GeneralLinearGroup(12,K);
D1:=DiagonalMatrix([t^5,t^3,t^10,t^12,t^0,t^2,t^4,t^6,t^6,t^6,t^6,t^6]);
D2:=DiagonalMatrix([u^8,u^8,u^8,u^8,u^8,u^8,u^8,u^8,u^6,u^4,u^2,u^7]);
G:=sub<GL12K | D1,D2>;
Gmod:=GModule(G);
chi:=Character(Gmod);
X:=CharacterTable(G);
Decomposition(X,chi);
```

Software Packages Referenced

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