MATH 2500 - Swinarski
Midterm 3
November 13, 2009

No calculators, computers, or other electronic devices are permitted.

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No books, notes, or formula sheets are permitted.

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Make sure to show all work.

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You have 50 minutes to complete this exam.

Name : Solutions

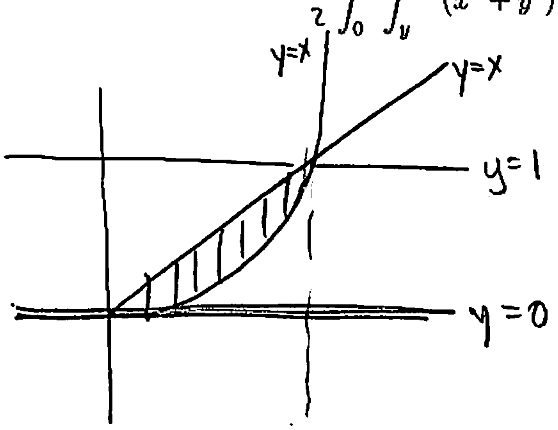
Good luck!

Question	Points	Score
1	13	
2	13	
3	4	
4	13	
5	26	
6	13	
7	13	
Total:	95	

1. (13 points) Draw a picture of the region in the x, y-plane described by the bounds of the following integral. Change the order of integration, and evaluate the integral.

$$\int_0^1 \int_y^{\sqrt{y}} (x^2 + y^2)\, dx dy$$

$$\int_0^1 \int_y^{\sqrt{y}} (x^2 + y^2)\, dx dy$$



$x = \sqrt{y}$

$x^2 = y$

$$\int_{x=0}^{1} \int_{y=x^2}^{x} (x^2+y^2)\, dy\, dx$$

$$= \int_{x=0}^{1} \left(x^2(x-x^2) + \tfrac{1}{3}x^3 - \tfrac{1}{3}x^6 \right) dx$$

$$= \int_{x=0}^{1} \left(x^3 - x^4 + \tfrac{1}{3}x^3 - \tfrac{1}{3}x^6\right) dx$$

$$= \int_{x=0}^{1} \left(\tfrac{4}{3}x^3 - x^4 - \tfrac{1}{3}x^6 \right) dx$$

$$\tfrac{1}{3}x^4 - \frac{x^5}{5} - \frac{x^7}{21} \Big|_0^1$$

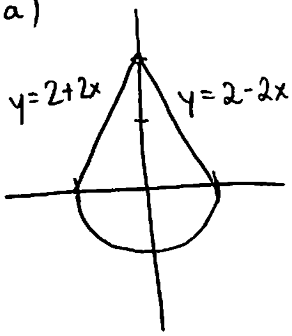
$$= \frac{1}{3} - \frac{1}{5} - \frac{1}{21} = \frac{3}{35}$$

2. (13 points) Suppose T is a thin plate enclosed on the bottom by the semicircle $y = -\sqrt{1-x^2}$ and above by the isosceles triangle with vertices $(-1,0), (1,0)$ and $(0,2)$. Suppose that the density of T is given by the function $\delta(x,y) = 3 - y$. Let $(\bar{x}, \bar{y})$ denote the center of mass.

(a) Draw a picture of T.

(b) Is it possible to determine either $\bar{x}$ or $\bar{y}$ just from the picture, without computing any integrals? If so, explain why.

a)



b) Yes, $\bar{x} = 0$ because both the region and the density function δ are symmetric about the y-axis

(c) Set up all the integrals needed to find the remaining coordinates of the center of mass of T. You do not need to evaluate these integrals.

$$M = \int_{x=-1}^{0} \int_{y=-\sqrt{1-x^2}}^{2+2x} (3-y)\,dy\,dx + \int_{x=0}^{1} \int_{y=-\sqrt{1-x^2}}^{2-2x} (3-y)\,dy\,dx$$

$M_y = 0$ by symmetry ($\bar{x}$ is on the y axis)

$$M_x = \int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{2+2x} y(3-y)\,dy\,dx + \int_{0}^{1} \int_{-\sqrt{1-x^2}}^{2-2x} y(3-y)\,dy\,dx$$

3. (4 points) Please supply the following formulas.

To switch between cylindrical and rectangular coordinates, one uses the formulas

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

and

$$dxdydz = r\, dr\, dz\, d\theta$$

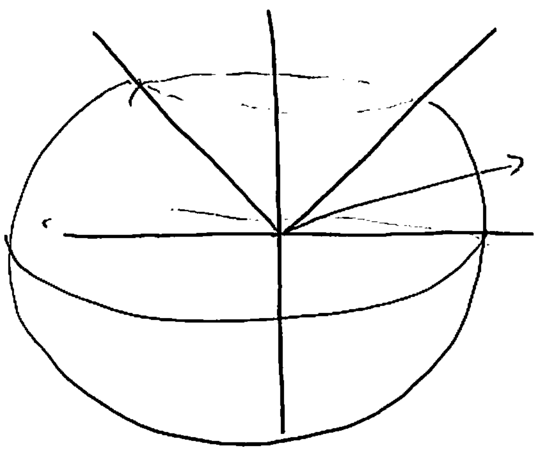
To switch between spherical and rectangular coordinates, one uses the formulas

$$x = \rho\cos\theta\sin\varphi$$
$$y = \rho\sin\theta\sin\varphi$$
$$z = \rho\cos\varphi$$

and

$$dxdydz = \rho^2\sin\varphi \quad d\rho\, d\varphi\, d\theta$$

4. (13 points) Let S be a sphere of radius 2 centered at the origin. Let R be the region which is inside S and below the cone $z = \sqrt{x^2 + y^2}$. Set up an integral in spherical coordinates to compute the volume of R. You do not need to evaluate it yet.



Where do the cone and sphere intersect?

$z = \sqrt{x^2+y^2}$

Sphere: $z = \sqrt{4-x^2-y^2}$

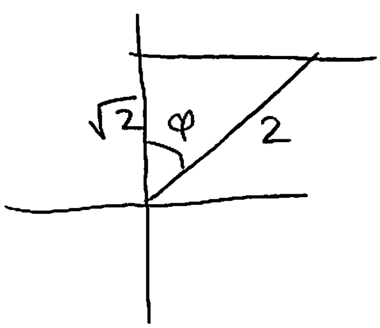
$\sqrt{x^2+y^2} = \sqrt{4-x^2-y^2}$

$x^2+y^2 = 4-x^2+y^2$

$2(x^2+y^2) = 4$

$x^2+y^2 = 2$

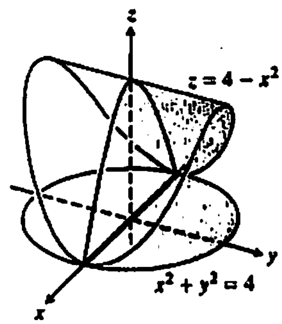
$z = \sqrt{2}$



Then $\varphi = 45^\circ$ or $\frac{\pi}{4}$

$$\text{Vol} = \int_{\theta=0}^{2\pi} \int_{\varphi=\frac{\pi}{4}}^{\pi} \int_{\rho=0}^{2} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

5. Let R be the region bounded below by the x, y-plane, on its sides by the cylinder $x^2 + y^2 = 4$, and above by the surface $z = 4 - x^2$. The region R is pictured below.



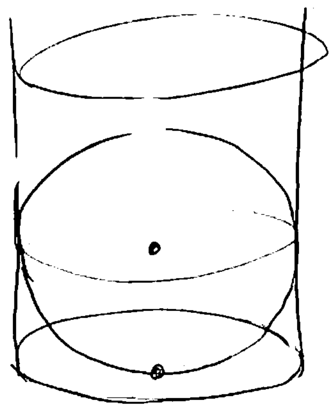
(a) (13 points) Set up an integral in rectangular coordinates to compute the volume of R. You do not need to evaluate it yet.

(b) (13 points) Set up an integral in cylindrical coordinates to compute the volume of R. You do not need to evaluate it yet.

a) $$\int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{4-x^2} dz\, dy\, dx$$

b) $$\int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{4-r^2\cos^2\theta} r\, dz\, dr\, d\theta$$

6. (13 points) A marble of radius 1cm is dropped into a cylindrical tube whose radius is also 1cm. The cylinder stands upright (like a graduated cylinder in chemistry) and is 20cm tall. The marble rests at the bottom of the cylinder. Set up an integral to compute how much water will fit inside the cylinder underneath the marble. (Hint: start by drawing a picture.) You may use whatever coordinates you think best (rectangular, cylindrical, or spherical). You do not need to evaluate the integral yet.



Let's draw a picture with the bottom of the cylinder in the x,y-plane. Then the center of the sphere is at $(0,0,1)$, so an equation for the sphere in rectangular coords. is $x^2+y^2+(z-1)^2=1$.

In cylindrical coords this is

$$r^2+(z-1)^2=1$$

or

$$z-1=\pm\sqrt{1-r^2}$$

$$z=\pm\sqrt{1-r^2}+1$$

We want the bottom half of the sphere as the upper boundary of the region

$$\text{Vol} = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{-\sqrt{1-r^2}+1} r\,dz\,dr\,d\theta$$

7. (13 points) Choose any one of the four integrals that you set up in Questions 4, 5, or 6, and evaluate it. You must choose only one — if you attempt several, I will grade the one you indicate, or the first one I see. I will not grade several attempts and give you the maximum points from all your attempts. Also, you should choose an integral that you are absolutely certain you set up correctly.

From #4.

$$\int_{\theta=0}^{2\pi} \int_{\varphi=\frac{\pi}{4}}^{\pi} \sin\varphi \int_{\rho=0}^{2} \rho^2 \, d\rho \, d\varphi \, d\theta$$

$$= \frac{8}{3} \int_{\theta=0}^{2\pi} \int_{\varphi=\frac{\pi}{4}}^{\pi} \sin\varphi \, d\varphi \, d\theta$$

$$-\cos\varphi \Big|_{\pi/4}^{\pi}$$

$$= \frac{8}{3} \int_{\theta=0}^{2\pi} \left(1 + \frac{\sqrt{2}}{2}\right) d\theta$$

$$= \frac{16\pi}{3}\left(1 + \frac{\sqrt{2}}{2}\right)$$

#5a.

$$\int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{4-x^2} dz \, dy \, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-x^2) \, dy \, dx$$

$$= \int_{-2}^{2} (4-x^2) \left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \right] dx$$

$$= \int_{-2}^{2} (4-x^2) \, 2\sqrt{4-x^2} \, dx$$

$$= 8\int_{-2}^{2} \sqrt{4-x^2} \, dx \; - \; 2\int_{-2}^{2} x^2\sqrt{4-x^2} \, dx$$

$\uparrow$

Not so easy to compute...

\# 5b.

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{2} r \int_{z=0}^{4-r^2\cos^2\theta} dz\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \left(4r - r^3\cos^2\theta\right) dr\, d\theta$$

$$2r^2 - \frac{1}{4}r^4\cos^2\theta \Big|_0^2$$

$$= \int_{\theta=0}^{2\pi} 8 - 4\cos^2\theta \, d\theta$$

$$= 8\theta - 4\left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right)\Big|_0^{2\pi}$$

$$= 16\pi - 4\pi = 12\pi$$

\# 6.

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{1} r \int_{z=0}^{-\sqrt{1-r^2}+1} dz\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_0^1 \left(-r\sqrt{1-r^2} + r\right) dr\, d\theta$$

$$\frac{1}{3}(1-r^2)^{3/2} + \frac{r^2}{2}\Big|_0^1$$

$$\left(0 + \frac{1}{2}\right) - \left(\frac{1}{3} - 0\right)$$

$$= \int_{\theta=0}^{2\pi} \frac{1}{6}\, d\theta = \frac{\pi}{3}$$