MATH 2500 - Swinarski
Midterm 2
October 14, 2009

No calculators, computers, or other electronic devices are permitted.
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No books, notes, or formula sheets are permitted.
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Make sure to show all work.
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You have 50 minutes to complete this exam.

Name : Solutions

Good luck!

Question	Points	Score
1	8	
2	7	
3	15	
4	20	
5	20	
6	20	
Total:	90	

1. (8 points) Let S be the explicit surface given by the equation

$$z = e^{x^2-y^2}.$$

Find an equation of the tangent plane of S through the point $(1,-1,1)$.

$$P = (1,-1,1)$$

$$\vec{n} = \left(\frac{\partial f}{\partial x}(1,-1), \frac{\partial f}{\partial y}(1,-1), -1 \right)$$

$$\frac{\partial f}{\partial x} = e^{x^2-y^2} \cdot (2x)$$

$$\frac{\partial f}{\partial y} = e^{x^2-y^2} \cdot (-2y)$$

$$\frac{\partial f}{\partial x}(1,-1) = 2$$

$$\frac{\partial f}{\partial x}(1,-1) = 2$$

$$\vec{n} = (2, 2, -1)$$

$$\vec{n} \cdot \left((x,y,z) - P \right) = 0$$

$$(2, 2, -1) \cdot (x-1, y-1, z-1) = 0$$

$$2x - 2 + 2y - 2 - z + 1 = 0$$

$$\boxed{2x + 2y - z = -1}$$

2. (7 points) Let S be the implicit surface given by the equation

$$xe^y \cos z - z = 1$$

Find an equation of the tangent plane of S through the point $(1, 0, 0)$.

$$\vec{n} = \nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)$$

$$\frac{\partial F}{\partial x} = e^y \cos z \qquad \frac{\partial F}{\partial x}(1,0,0) = 1$$

$$\frac{\partial F}{\partial y} = x \cos z \, e^y \qquad \frac{\partial F}{\partial y}(1,0,0) = 1$$

$$\frac{\partial F}{\partial z} = xe^y(-\sin z) - 1 \qquad \frac{\partial F}{\partial z}(1,0,0) = -1$$

$$\vec{n} = (1, 1, -1)$$

$$(1, 1, -1) \cdot (x-1, y, z) = 0$$

$$x - 1 + y - z = 0$$

$$\boxed{x + y - z = 1}$$

3. Let P be the point $(2,4)$. Let $f(x,y)$ be a function. Suppose $\nabla f(2,4) = (5,12)$.

(a) (4 points) What unit vector $\mathbf{u}$ gives the largest directional derivative $D_{\mathbf{u}}f(2,4)$? (Remember, your answer should be a unit vector.)

When $\vec{u}$ is in direction of gradient

$$\vec{u} = \frac{(5,12)}{\|(5,12)\|} = \left(\frac{5}{13}, \frac{12}{13}\right)$$

(b) (4 points) When $\mathbf{u}$ is your answer from part (a), what is $D_{\mathbf{u}}f(2,4)$?

$$D_{\vec{u}} f = \vec{u} \cdot \nabla f$$

$$= \left(\frac{5}{13}, \frac{12}{13}\right) \cdot (5, 12)$$

$$= \frac{169}{13} = 13$$

(c) (2 points) Suppose that $x = 2t - 2$ and $y = t^2$. What value of t gives $P = (2,4)$?

When $t = 2$, $(x,y) = (2,4)$

(d) (5 points) Use the Chain Rule to compute $\frac{df}{dt}(2)$.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \nabla f = (5,12) \text{ at } (2,4)$$

$$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = 2t \qquad \frac{dy}{dt}(t=2) = 4$$

$$\frac{df}{dt}(2) = (5)(2) + (12)(4) = 10 + 48 = 58$$

4. (20 points) *The second derivative test.* Let

$$f(x,y) = \frac{1}{3}x^3 - xy + x + \frac{1}{3}y^3 - y^2 + y - 1.$$

Find all the critical points of f. For each critical point, use the second derivative test to determine whether the critical point is a local maximum, a local minimum, or a saddle point, or that the test is inconclusive.

Solve $\nabla f = (0,0)$: $\nabla f = (x^2 - y + 1, -x + y^2 - 2y + 1)$

$x^2 = y - 1$ $\qquad$ $x = y^2 - 2y + 1 = (y-1)^2$

Substitute the 2nd equation into the 1st :

$$x^2 = ((y-1)^2)^2 = y - 1$$

$$(y-1)^4 = y - 1$$

$$(y-1)((y-1)^3 - 1) = 0$$

$$\Rightarrow y = 1 \quad \text{or} \quad (y-1)^3 = 1$$

$$y = 2$$

If $y = 1$ then $x = (y-1)^2 = 0$

If $y = 2$ then $x = (y-1)^2 = 1$

Get 2 critical points, $(0,1)$ and $(1,2)$

Second Derivative Test :

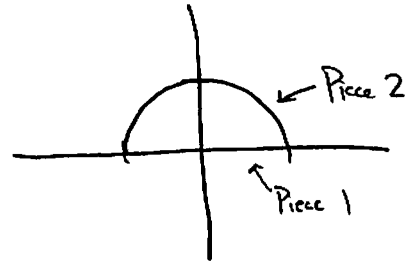
$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2x & -1 \\ -1 & 2y - 2 \end{pmatrix}$$

Critical point	f_{xx}	f_{yy}	f_{xy}	$f_{xx} f_{yy} - f_{xy}^2$	Concavity	Descriptions
$(0,1)$	0	0	-1	-1		saddle point
$(1,2)$	2	2	-1	$4-1=3$	concave up	local min.

5. (20 points) Let D be the upper half of the unit disk in the xy-plane centered at $(0,0)$. Let

$$f(x,y) = 2x^3 + y^4.$$

Find the absolute minima and maxima of f on D.



① Look for interior critical points

$$\nabla f = (6x^2, 4y^3) = (0,0)$$

Only one critical point, $(0,0)$ (and it's actually on the boundary)

② Check each boundary piece

Piece 1: $y=0$

$$f\big|_{y=0} = 2x^3$$

$$\frac{d}{dx} = 6x^2$$

$\frac{d}{dx} = 0$: $x = 0$ Get one critical point, $(0,0)$

Piece 2: $y = \sqrt{1-x^2}$

$$f\big|_{y=\sqrt{1-x^2}} = 2x^3 + (1-x^2)^2$$

$$= 2x^3 + 1 - 2x^2 + x^4$$

$$\frac{d}{dx} = 6x^2 - 4x + 4x^3$$

$\frac{d}{dx} = 0$:

$$4x^3 + 6x^2 - 4x = 0$$

$$2x(2x^2 + 3x - 2) = 0$$

$\Rightarrow$ $x = 0$ or $2x^2 + 3x - 2 = 0$

$$x = \frac{-3 \pm \sqrt{9 + 4\cdot 2\cdot 2}}{4}$$

$$= \frac{-3 \pm 5}{4}$$

$$= -2, \frac{2}{4}$$

continued →

Piece 2, continued:

If $x = 0$, then $y = \sqrt{1-x^2} = 1 \Rightarrow (0,1)$

If $x = -2$: not in D

If $x = \frac{1}{2}$ then $y = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2} \Rightarrow \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

List of points to check:

	(x,y)	$f(x,y)$	
critical points from interior and edges	$(0,0)$	0	
	$(0,1)$	1	
	$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$	$2 \cdot \frac{1}{8} + \frac{9}{16} = \frac{13}{16}$	
Points where boundary pieces hit each other	$(1,0)$	2	← abs. max
	$(-1,0)$	-2	← abs min

6. (20 points) *Lagrange multipliers.* Use Lagrange multipliers to find the maxima and minima of the function

$$f(x, y) = 2x^2 + 3y^2 - 4x - 5$$

subject to the constraint

$$x^2 + y^2 - 16 = 0.$$

Note: there are other valid methods for solving this problem, but to get credit, you must use Lagrange multipliers.

$\nabla f = \lambda \nabla g$ $\qquad$ $\nabla f = (4x-4, 6y)$

$\nabla g = (2x, 2y)$

System of equations:

$$4x - 4 = 2x\lambda$$
$$6y = 2y\lambda$$
$$x^2 + y^2 = 16$$

To solve: 2^{nd} equation says $\quad 6y - 2y\lambda = 0$

$$2y(3-\lambda) = 0$$

$\Rightarrow$ either $y = 0$ or $\lambda = 3$

If $y = 0$: use the equation $x^2 + y^2 = 16$

Then $x^2 = 16$

so $x = \pm 4$

Get 2 points, $(4, 0)$, $(-4, 0)$

If $\lambda = 3$: Then the 1^{st} equation says

$$4x - 4 = 6x$$
$$-4 = 2x$$
$$x = -2$$

Then using $x^2 + y^2 = 16$, get

$$y^2 = 16 - x^2 = 16 - 4 = 12$$
$$y = \pm\sqrt{12}$$

List of points to check:

(x,y)	$f(x,y)$	
$(4,0)$	$32 - 16 - 5 = 11$	← abs min
$(-4,0)$	$32 + 16 - 5 = 43$	
$(-2, \sqrt{12})$	$8 + 36 + 8 - 5 = 47$	
$(-2, -\sqrt{12})$	$= 47$	← abs max