MATH 2500 - Swinarski
Midterm 1
September 25, 2009

No calculators, computers, or other electronic devices are permitted.

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No books, notes, or formula sheets are permitted.

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Make sure to show all work.

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You have 50 minutes to complete this exam.

Name : Solutions

Good luck!

Question	Points	Score
1	10	
2	12	
3	24	
4	24	
5	12	
6	13	
Total:	95	

1. (10 points) Let $P = (3, 10, 9)$, $Q = (-5, 4, 3)$, $R = (9, 8, 2)$. Find an equation of the plane through P, Q, R. Please write your answer in the form $ax + by + cz = d$.

$$\vec{PQ} = Q - P = (-8, -6, -6)$$

$$\vec{PR} = R - P = (6, -2, -7)$$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -8 & -6 & -6 \\ 6 & -2 & -7 \end{vmatrix}$$

$$= (30, -92, 52)$$

$$\vec{n} \cdot ((x, y, z) - P) = 0$$

$$(30, -92, 52) \cdot (x - 3, y - 10, z - 9) = 0$$

$$30x - 90 + 92y + 920 + 52z - 468 = 0$$

$$30x + 92y + 52z = -362$$

$$\boxed{15x - 46y + 26z = -181}$$

2. A tetrahedron is a pyramid with four vertices, six edges, and four faces. The edges all have equal length. In particular, each face is an equilateral triangle, and the faces are all congruent to each other.

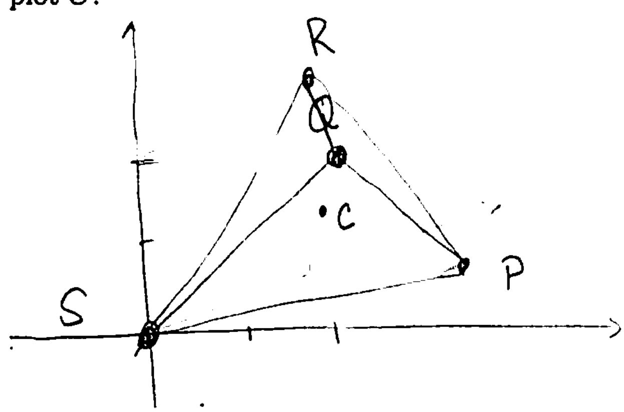
   Let's find the central angle in a tetrahedron. This is important in chemistry, where the bonds around a carbon atom tend to a tetrahedral configuration.

   We can create a tetrahedron as follows. The vertices are

$$\begin{aligned} P &= (2,2,0) \\ Q &= (2,0,2) \\ R &= (0,2,2) \\ S &= (0,0,0) \end{aligned}$$

   Then $C = (1,1,1)$ is the central point of the tetrahedron.

   (a) (3 points) Plot the points P, Q, R, S and the line segments between them. Also plot C.



   (b) (2 points) Find the vector **u** between C and P.

$$\vec{u} = C - P = (1,1,1) - (2,2,0) = (-1,-1,1)$$

   (c) (2 points) Find the vector **v** between C and Q.

$$\vec{v} = C - Q = (1,1,1) - (2,0,2) = (-1,1,-1)$$

   (d) (5 points) What is the angle between **u** and **v**?

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{-1}{\sqrt{3}\,\sqrt{3}} = -\frac{1}{3}$$

$$\boxed{\theta = \arccos\left(-\frac{1}{3}\right)}$$

3. Let $\mathbf{r}(t)$ be the parametrized curve

$$\mathbf{r}(t) = (\frac{2}{3}t^3, 1, t^2).$$

Let $P = (0,1,0)$ and $Q = (\cancel{0}\ \tfrac{2}{3}, 1, 1)$.

(a) (12 points) Find the arclength of $\mathbf{r}(t)$ between P and Q.

$$L = \int_{t=a}^{b} \|\vec{r}\,'(t)\|\, dt \qquad P = \vec{r}(0) \quad Q = \vec{r}(1)$$

$$\vec{r}\,'(t) = (2t^2, 0, 2t)$$

$$\|\vec{r}\,'(t)\| = \sqrt{4t^4 + 4t^2} = \sqrt{4t^2(t^2+1)} = |2t|\sqrt{t^2+1}$$

but can drop abs. value because t is between 0 and 1

$$L = \int_{t=0}^{1} 2t\sqrt{t^2+1}\, dt$$

$u = t^2+1 \quad du = 2t\, dt \qquad t=0 \Rightarrow u=1 \quad t=1 \Rightarrow u=2$

$$= \int_{u=1}^{2} u^{1/2}\, du = \frac{2}{3}u^{3/2}\Big|_1^2$$

$$= \boxed{\frac{2}{3}\sqrt{8} - \frac{2}{3}}$$

(b) (12 points) Find an equation of the tangent line at Q.

$$\vec{v} = \vec{r}\,'(1) \qquad \vec{r}\,'(t) = (2t^2, 0, 2t) \quad \vec{r}\,'(1) = (2, 0, 2)$$

$$\vec{R}(T) = Q + T\vec{v}$$

$$= (\tfrac{2}{3}, 1, 1) + T(2, 0, 2)$$

$$= (\tfrac{2}{3} + 2T, 1, 1 + 2T)$$

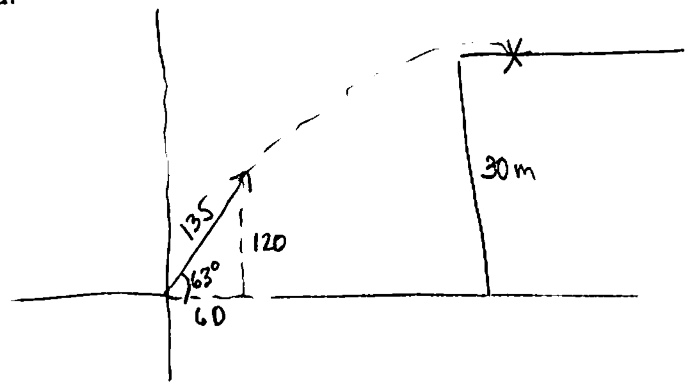
4. It's June 6, 1944: D-Day. You are a gunner with the US infantry, and your orders are to attack an enemy position on the cliffs 30 meters above Omaha beach.

   You set your 50mm M2 mortar to an angle of 63 deg. It fires with a muzzle velocity of 135 meters/second. An approximation to the initial velocity vector is (60, 120) meters/second.

   Also, you may use the approximation $g \approx 10$ for the questions below.

   (a) (12 points) Write an equation $\mathbf{r}(t)$ for the position of your shell at time t after it is fired.

The problem doesn't specify a starting point. Hence we are free to choose where to draw the coordinate axes. I chose to make $\vec{r}(0) = (0,0)$ in my picture.



Assume $\vec{a}(t) = (0, -10)$

$$\vec{v} = \int \vec{a}(t)\,dt = (c_1, -10t + c_2)$$

$$\vec{v}(0) = (60, 120) = (c_1, -10 \cdot 0 + c_2)$$

$$\Rightarrow c_1 = 60, \quad c_2 = 120$$

$$\vec{v}(t) = (60, -10t + 120)$$

$$\vec{r}(t) = \int \vec{v}(t)$$

$$= (60t + c_3, -5t^2 + 120t + c_4)$$

$$\vec{r}(0) = (0,0) \Rightarrow c_3 = 0, c_4 = 0$$

$$\boxed{\vec{r}(t) = (60t, -5t^2 + 120t)}$$

(b) (12 points) How long will it take for the shell to hit the enemy position?

We know that when the shell hits, its y-value matches the enemy position's y-value

So solve

$$-5t^2 + 120t = 30$$

$$0 = 5t^2 - 120t + 30$$

$$0 = 5\left(t^2 - 24t + 6\right)$$

$$t = \frac{24 \pm \sqrt{24^2 - 24}}{2}$$

5. (12 points) Let S be the explicit surface

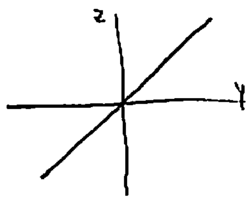
$$z = ye^x$$

Sketch a graph of S.

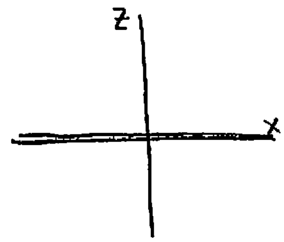
Let's graph cross sections.

e^x is never 0 (look at its graph) so if $z=0$ we must have $y=0$

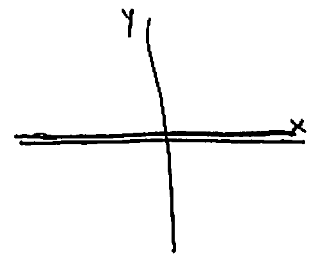
$\boxed{x=0}$ Get $z=y$



$\boxed{y=0}$ Get $z=0$

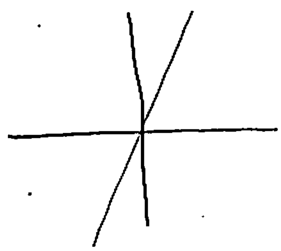


$\boxed{z=0}$



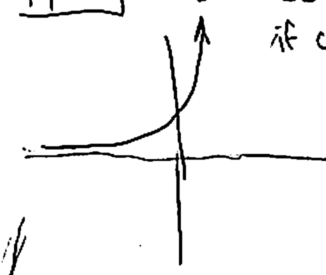
Hard to see the surface from just these cross sections. Try a few more:

$\boxed{x=c}$ $z = e^c y$

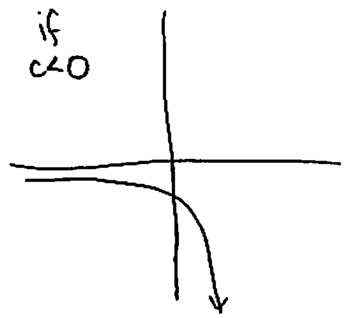


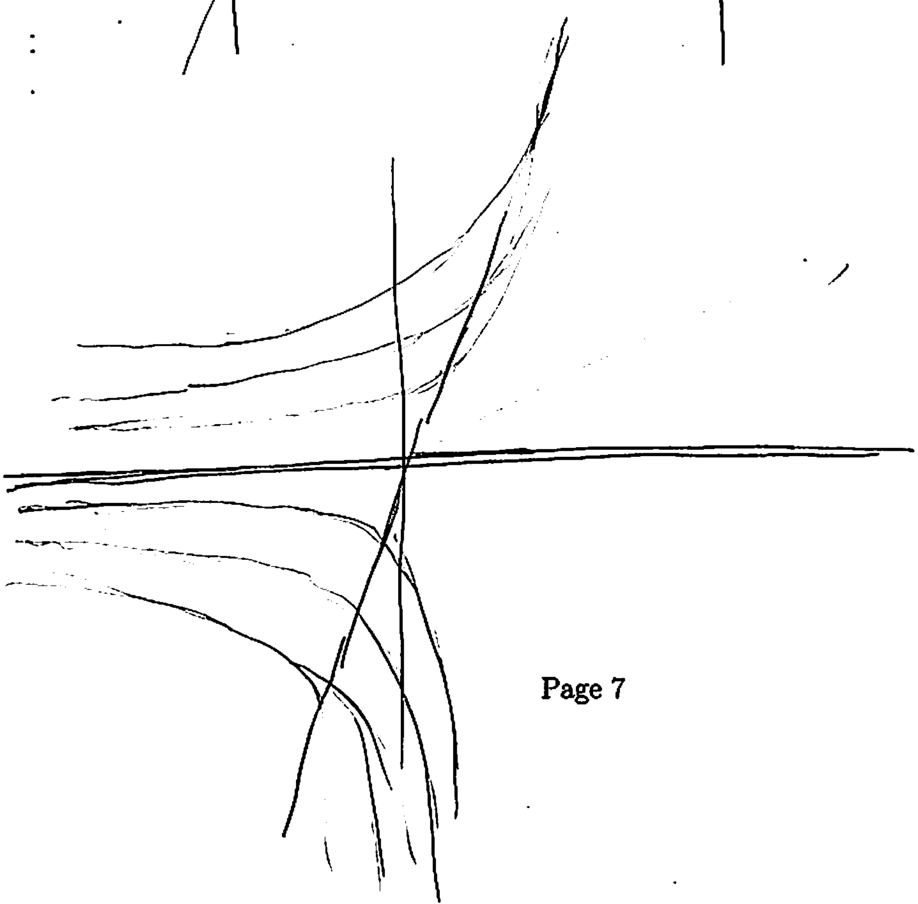
$\boxed{y=c}$ $z = ce^x$

if $c>0$



if $c<0$





6. Let $f(x,y)$ be a function of two variables. Let $P=(a,b)$ be a point in the

(a) (2 points) Give the formal definition of the partial derivative $\frac{\partial f}{\partial x}(a,b)$.

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$ ✓

(b) (2 points) Give the formal definition of the partial derivative $\frac{\partial f}{\partial y}(a,b)$.

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$ ✓

(c) (3 points) Give a short explanation (one or two sentences) explaining what the partial derivative means.

The partial derivative $\frac{\partial f}{\partial x}(a,b)$ is the slope of the tangent line through $(a, f(a,b))$ of the curve which is the cross section of $z = f(x,y)$ with the plane $y = b$.

(d) (3 points) Let $f(x,y) = \sin(x^2y)$. Find $\frac{\partial f}{\partial x}$ as a function of x and y.

$$\frac{\partial f}{\partial x} = \cos(x^2y) \cdot 2xy$$

(e) (3 points) Let $f(x,y) = \sin(x^2y)$. Find $\frac{\partial f}{\partial y}$ as a function of x and y.

$$\frac{\partial f}{\partial y} = \cos(x^2y) \cdot x^2$$