State polytopes and Geometric Invariant Theory

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This is joint work with Ian Morrison (Fordham University).

A preprint is available on my website:

`http://www.uga.edu/~davids/research.html`

We plan to post it on the arXiv later this summer after we add a few more examples.

Introduction

Let M_g denote the moduli space of smooth projective curves of genus g.

It has been studied for over 150 years, but there are still many open questions about its geometry and topology.

M_g has a very nice compactification, denoted $\overline{M}_g$. The boundary $\Delta = \overline{M}_g \setminus M_g$ consists of nodal curves with finite automorphism groups.

In the 1970s Mumford, Gieseker, and Knudsen proved that $\overline{M}_g$ is a projective variety. We would like to know more about its birational geometry.

Recall Mumford and Gieseker's construction of M_g via Geometric Invariant Theory (GIT):

Let C be a smooth curve of genus g.

Fix an integer $\nu \geq 5$. Consider the embedding $C \to P^N$ associated to the complete linear system $|H^0(C, \omega_C^{\otimes \nu})|$.

Then C is represented by a $PGL(N+1)$ orbit in $Hilb := Hilb(\mathbf{P}^N, dt - g + 1)$.

IDEA: find the locus $J \subset Hilb$ parametrizing smooth curves C with

$$\mathcal{O}_{\mathbf{P}^N}(1)|_C \cong \omega_C^\nu$$

Then, set-theoretically, we expect $J/SL(N+1) = M_g$.

Constructing quotients in algebraic geometry is sometimes complicated.

- We need additional data: a linearization of the $SL(N+1)$-action on J.

  We usually specify this by describing an equivariant morphism $J \to \mathbf{P}^R$ and pulling back $\mathcal{O}_{\mathbf{P}^R}(1)$.
- Also, the GIT quotient may not be the set-theoretic quotient.

  More orbits may be identified:

$$x \sim y \text{ if } \mathrm{Orb}(x) \cap \overline{\mathrm{Orb}(y)} \neq \emptyset \text{ or } \overline{\mathrm{Orb}(x)} \cap \mathrm{Orb}(y) \neq \emptyset.$$

Theorem (Mumford, Gieseker)

If $\nu \geq 5$, and if the morphism $J \to \mathbf{P}^R$ is the restriction of an embedding $J \subset Hilb \hookrightarrow Gr \hookrightarrow \mathbf{P}^R$ of sufficiently high degree, then $J//SL(N+1) \cong M_g$, and $\overline{J}//SL(N+1) \cong \overline{M}_g$.

QUESTION: What if we allow ν to vary, and/or allow the linearization to vary?

We expect that smooth curves are likely to be GIT stable for many degrees, linearizations, so many GIT quotients will be birational to $\overline{M}_g$.

OBSERVATION (Hassett, Hyeon, Lee): Frequently, spaces which arise in the Log Minimal Model Program for $(\overline{M}_g, \Delta)$ are GIT quotients.

THEME : Mumford and Gieseker's theoretical techniques are successful for high degree embeddings and linearizations.

Their techniques break down for small ν and low degree linearizations.

But in low degrees, we can attempt a more computational approach.

GIT for Hilbert schemes

Let k be an algebraically closed field of characteristic 0.

Let $S = k[x_0, \ldots, x_N]$. Write S_m for the degree m polynomials.

Let $I \subset S$ be a graded ideal defining a projective scheme in $\mathbf{P}^N$.

Let I_m be the degree m slice of I.

Let φ denote the Hilbert function: $\varphi(m) = \dim I_m$.

Then $I_m \subset S_m$ determines a point of the Grassmannian $Gr\left(\binom{m+N}{m}, \varphi(m)\right)$

We can write the point $I_m \in Gr$ very explicitly:

1. Choose ordered bases $\mathcal{B}_{S_m}$ and $\mathcal{B}_{I_m}$ of S_m and I_m.
2. Write a matrix M as follows:
   - Label columns by $\mathcal{B}_{S_m}$
   - Label rows by $\mathcal{B}_{I_m}$
   - $M_{i,j}$ is the coefficient of the j^{th} monomial in the i^{th} generator
3. The Plücker coordinates of I_m are given by the $\varphi(m) \times \varphi(m)$ minors of M.

EXAMPLE: two points in $\mathbf{P}^2$. Say $P = (1,2,3)$ and $Q = (5,1,-4)$. Write $\mathbf{P}^2 = \mathrm{Proj}(k[a,b,c])$. Then

$$\begin{aligned} I &= (c-3a, b-2a) * (a-5b, c+4b) \\ &= (3a^2 - 15ab - ac + 5bc, 12ab + 3ac - 4bc - c^2, \\ &\quad 2a^2 - 11ab + 5b^2, 8ab - 4b^2 + 2ac - bc) \end{aligned}$$

Let $\mathcal{B}_{S_2} = \{a^2, ab, ac, b^2, bc, c^2\}$. Then M is given by

		a^2	ab	ac	b^2	bc	c^2
f_1	:	3	−15	−1	0	5	0
f_2	:	0	12	3	0	−4	−1
f_3	:	2	−11	0	5	0	0
f_4	:	0	8	2	−4	−1	0

and the Plücker point of M is

$[45 : -95 : 99 : -154 : 209 : 55 : -18 : 38 : -13 : -83 : 108 : -228 : 22 : 55 : -132]$

More GIT: The numerical criterion

Mumford's numerical criterion: to study GIT stability of a point $x \in \mathbf{P}^N$ with respect to the action of $SL(N+1)$ and the linearization on $\mathcal{O}_{\mathbf{P}^N}(1)$, we may work with one 1-PS of $SL(N+1)$ at a time:

Criterion

Let λ be a 1-PS of $SL(N+1)$ acting with weights r_i. Let x^* be a point lying over x. Compute

$$\mu^L(x, \lambda) := \max\{-r_i \mid i \textit{ such that } x_i^* \neq 0\}.$$

Then

$$x \textit{ is } \lambda\textit{-stable} \iff \mu < 0.$$
$$x \textit{ is } SL(N+1) \textit{ stable} \iff x \textit{ is } \lambda\textit{-stable for all } \lambda \textit{ in } SL(N+1).$$

$$\boxed{\mu^L(x,\lambda) := \max\{-r_i \mid i \text{ such that } x_i^* \neq 0\}.}$$

$$\boxed{x \text{ is } \lambda\text{-stable} \iff \mu < 0.}$$

Equivalently: Let T be a maximal torus in $SL(N+1)$. Then

x is T-stable $\iff$
for every 1-PS λ, x has a nonzero coordinate where λ acts with positive weight and x has nonzero coordinate where λ acts with negative weight

Given x, how can we find coordinates with positive/negative weights?

Special case: Let T be the maximal torus scaling the variables a, b, c.

Let λ be a 1-PS in T acting with weights t, u, v on the variables a, b, c. Hence λ acts on a monomial $m = a^x b^y c^z$ with weight $(t, u, v) \cdot (x, y, z)$.

On a simple wedge product $m_1 \wedge \cdots \wedge m_k$, the 1-PS λ has weight

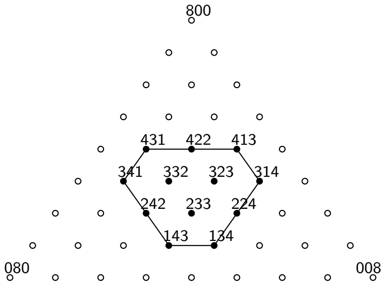
$$(t, u, v) \cdot (x_1, y_1, z_1) + (t, u, v) \cdot (x_2, y_2, z_2) + \cdots + (t, u, v) \cdot (x_k, y_k, z_k)$$

$$= (t, u, v) \cdot \left(\sum_{i=1}^{k} x_i, \sum_{i=1}^{k} y_i, \sum_{i=1}^{k} z_i\right).$$

Thus, exponent vectors may be identified with characters, and we may simplify our calculations by working with the set of characters, which is smaller than the set of Plücker coordinates.

In our example, 15 Plücker coordinates collapse to 12 exponent vectors.

Example: $\widehat{12} = ac \wedge b^2 \wedge bc \wedge c^2 \mapsto (1, 3, 4)$.



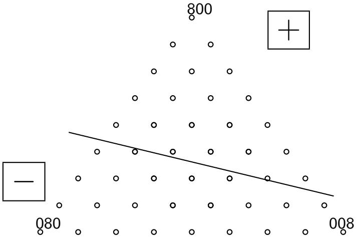
Definition

The m^{th} *state* of I is the set of characters where M has a nonzero Plücker coordinate.
The m^{th} *state polytope* of I is the convex hull of the state.

The characters all lie on a hyperplane in $\mathbb{Z}^3$. In this example, $x + y + z = 8$ in $\mathbb{Z}^3$. We call the point $(\frac{8}{3}, \frac{8}{3}, \frac{8}{3})$ the barycenter of this hyperplane.

Recall: the λ-weight on a character (x, y, z) is obtained just by dotting $(t, u, v) \cdot (x, y, z)$. Draw the hyperplane $(t, u, v) \cdot (x, y, z) = 0$. Then this separates the characters according to whether they have positive or negative λ-weight.

Example: (t,u,v) = (17, -13, -4)



We have the following characterization of GIT stability:

Criterion

x is T-stable iff the barycenter lies in the interior of the state polytope. x is T-strictly semistable iff the barycenter lies on the boundary of the state polytope.

Goals:

1. Find a fast way to compute states and state polytopes.
2. Extend this criterion from T-stability to $SL(N+1)$-stability.

Monomial ideals

SPECIAL CASE: Suppose I is a monomial ideal. Recall the definition of the matrix M:

1. Choose ordered bases $\mathcal{B}_{S_m}$ and $\mathcal{B}_{I_m}$ of S_m and I_m.
2. $M_{i,j}$ is the coefficient of the j^{th} monomial in the i^{th} generator
3. The Plücker coordinates of I_m are given by the $\varphi(m) \times \varphi(m)$ minors of M.

But if I is a monomial ideal, then the matrix M will have exactly one nonzero $\varphi(m) \times \varphi(m)$ minor.
Thus, I has only one nonzero Plücker coordinate, and the state of I is a single point.

Gröbner fans

DEFINITIONS :
A multiplicative term order $\preceq$ on S is a set of total orders, one for each degree m, on the monomials of degree m.

These must satisfy a compatibility condition:
if $M_i \preceq M_j$ in degree m, and N is any other monomial of degree n, then $M_i N \preceq M_j N$ in degree $m + n$.

EXAMPLE: the lexicographic (or alphabetic) order with $x \preceq y \preceq z$.
Here $x^3y^2z \preceq x^3yz^2$ since *xxxyyz* would come before *xxxyzz* in an English dictionary.

Let $f = \sum_{j \in A} a_j M_j$, where $a_j \in k$ and each M_j is a monomial. If $L \in I$ satisfies $M_L \preceq M_j$ for all $j \in I \setminus \{L\}$, then we call M_L the leading term of f, and write $M_L =: \text{in}_{\preceq} f$.

The initial ideal of I with respect to $\preceq$ is

$$\text{in}_{\preceq} I := (\text{in}_{\preceq} f \mid f \in I).$$

Let $(f_1, \ldots, f_k)$ be a set of generators for I. In general,

$$\mathrm{in}_{\preceq} I \neq (\mathrm{in}_{\preceq} f_1, \mathrm{in}_{\preceq} f_2, \ldots, \mathrm{in}_{\preceq} f_k)$$

So if a set of generators $(g_1, \ldots, g_s)$ of I has the special property that

$$\mathrm{in}_{\preceq} I = (\mathrm{in}_{\preceq} g_1, \mathrm{in}_{\preceq} g_2, \ldots, \mathrm{in}_{\preceq} g_s)$$

we call $(g_1, \ldots, g_s)$ a Gröbner basis for I with respect to $\preceq$.

If no nonzero monomial in g_i is divisible by $\mathrm{in}_{\preceq} g_j$ for $j \neq i$, then we call $(g_1, \ldots, g_s)$ a reduced Gröbner basis with respect to $\preceq$.

Fix a multiplicative term order $\preceq$.

Proposition

1. *Every nonzero ideal I has a unique reduced Gröbner basis with respect to $\preceq$.*
2. *A reduced Gröbner basis is characterized by its initial ideal.*

There are infinitely many term orders on S. However, two different term orders $\preceq_1$ and $\preceq_2$ might determine the same initial ideal for I.

Proposition

Fix I. Then there are only finitely many distinct initial ideals for I, as $\preceq$ varies over all possible term orders.

The application to GIT comes from the following theorem:

Theorem (Bayer–Morrison, 1981)

Suppose that m is chosen sufficiently large that the map

$$\begin{array}{rcl} Hilb & \longrightarrow & Gr \\ I & \mapsto & I_m \subset S_m \end{array}$$

is an embedding. Then there is a bijection

$$\{\textit{vertices of } \mathrm{State}_m(I)\} \leftrightarrow \{\textit{initial ideals of } I\}$$

QUESTION: Is there a fast way to compute the set of initial ideals of an ideal?

Answer: Yes!

There is a free, open source program called `gfan`, written by Anders Jensen, which computes the set of initial ideals of I.

So to compute a state polytope, we may

1. Use `gfan` to compute the set of initial ideals of I.
2. Transform each initial ideal to a vertex of the state polytope using the bijection from the Bayer–Morrison theorem.

I have implemented this in the `Macaulay 2` package `StatePolytope`.

EXAMPLE: Let's compute the state polytope of our two points in $\mathbf{P}^2$.

```
> M2
i1 : loadPackage("StatePolytope")
i2 : R = QQ[a,b,c];
i3 : I1 = ideal(c-3*a,b-2*a);
i4 : I2 = ideal(a-5*b,c+4*b);
i5 : I = I1*I2;
i6 : initialIdeals(I)
o6 : List
o6 : {{b^2, a*c, a*b, a^2}, {c^2, a*c, a*b, a^2},
      {c^2, b*c, a*c, a^2}, {c^2, b*c, b^2, a*c},
      {b*c, b^2, a*b, a^2}, {c^2, b*c, b^2, a*b}}
```

The Bayer–Morrison bijection:

Given an initial ideal I,

to find the corresponding vertex of $\text{State}_m(I)$,

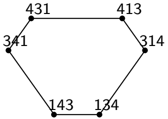
find all the monomials of degree m inside the initial ideal, and add their exponent vectors together.

Example : $I = (b^2, ac, ab, a^2), m = 2$. Then the degree 2 monomials in this initial ideal are just the generators themselves, and the sum of their exponent vectors is $(4, 3, 1)$.

We complete the calculation for two points in $\mathbf{P}^2$:

```
i7 : statePolytope(2,I)
o7 = {{4, 3, 1}, {4, 1, 3},
      {3, 1, 4}, {1, 3, 4},
      {3, 4, 1}, {1, 4, 3}}
```

This matches our previous calculation:



When m is small

PROBLEM: One hypothesis in the Bayer–Morrison Theorem is that m must be large.

There is an explicit bound for how large m must be (the Gotzmann number), but we are interested in examples where m is smaller than this.

The map used in the proof of the Bayer–Morrison is not a bijection when m is small. It is easy to produce examples where two initial ideals give the same vertex of the state polytope.

However, even when m is small, we can still show that every initial ideal is in the state:

Proposition (Morrison–Swinarski)

$\mathrm{State}_m(\mathrm{in}_{\preceq} I) \in \mathrm{State}_m(I)$, *for any m.*

So we may still be able to use Gröbner techniques to prove that an ideal is T-stable.

Open Questions

In what ways can the Bayer–Morrison Theorem break down for small m?

1. *Can* $\mathrm{State}_m(\mathrm{in}_\preceq I)$ *ever fail to be a vertex of* $\mathrm{State}_m(I)$*? (Conjecture: no)*
2. *Can there be vertices of* $\mathrm{State}_m(I)$ *which are not of the form* $\mathrm{State}(\mathrm{in}_\preceq I)$*?*

When m is large

It is known that GIT stability for the Chow variety is closely related to GIT stability for the Hilbert scheme when $m >> 0$.

One can define a Chow state and a Chow polytope using ideas similar to those used for Hilbert schemes.

Kapranov, Sturmfels, and Zelevinsky proved that the Chow polytope is a scaled limit of (outer) state polytopes as $m \to \infty$. So with a little extra work, we can compute Chow polytopes and Chow stability, too.

Fix an initial ideal I. Recall that the state of I is just a point.

Instead of adding up the exponents of the monomials in I_m, we could add up all the exponents of the monomials outside I_m. I call this the outer state of I.

Let d be the largest degree of a generator of I. Then for all $m \geq d$, each coordinate of $\mathrm{OuterState}_m(I)$ is given by a polynomial of degree at most r, where r is the dimension of I (hence $\dim V(I) + 1$).

Example: $I = (b^2, ac, ab, a^2)$.

m	$\text{OuterState}_m(I)$
2	(0,1,3)
3	(0,1,5)
4	(0,1,7)

Interpolate in each coordinate:

$$\text{OuterState}_m(I) = (0, 1, 2m - 1)$$

Then the Chow point is $\lim_{m\to\infty}(\frac{0}{m}, \frac{1}{m}, \frac{2m-1}{m}) = (0, 0, 2)$.

Applying this to all six initial ideals for the two points in $\mathbf{P}^2$, we get that the Chow polytope is the triangle with vertices $(2, 0, 0), (0, 2, 0), (0, 0, 2)$.

This is typical behavior. The Chow polytope is often simpler than the state polytopes, because several vertices of $\text{State}_m(I)$ may come together in the limit.

Comparison to Hassett, Hyeon, and Lee's techniques

Hassett, Hyeon, and Lee have also developed on Gröbner techniques for GIT stability.

- They compute one Gröbner basis for one term order
- This tells them about stability for one 1-PS
- They use this to get precise information about the structure of GIT quotient map (which orbits get identified)

- We compute Gröbner bases for all possible term orders
- This tells us stability for all 1-PS in T

Hassett, Hyeon, and Lee also have Macaulay 2 functions. These have been very useful to us for GIT unstable examples. Often, we can guess a destabilizing 1-PS and check it with their `MUm` function, instead of computing the entire state polytope to see that it doesn't contain the barycenter.

Hassett, Hyeon, and Lee also proved a very useful theoretical result:

Proposition

Let $C \subset \mathbf{P}(V)$ be a projective variety, ρ a 1-PS of $SL(V)$, and C^* the variety to which $\rho(t).C$ specializes. Suppose that C and C^* satisfy:

- *C and C^* are connected and of pure dimension one;*
- *$V \to \Gamma(\mathcal{O}_C(1))$ and $V \to \Gamma(\mathcal{O}_{C^*}(1)))$ are isomorphisms*
- *$\mathcal{O}_C(1)$ and $\mathcal{O}_{C^*}(1)$ are 2-regular.*

Then

$$\mu_m(C,\rho) = (m-1)[(3-m)\mu_2(C,\rho) + (m/2-1)\mu_3(C,\rho)]$$

Frequently we have been able to compute stability for just a few small values of m, and apply this result to conclude stability for all m.

PROBLEM: All the results so far have been for T-stability. That is, we have only worked with 1-PS λ in T, not all 1-PS in $SL(N+1)$.

From T to G: Multiplicity freeness

We have identified a class of special examples for which T-stability implies $SL(R+1)$-stability.

Definition

Let C be a projective scheme, L an ample line bundle. We say that (C, L) is multiplicity free if the action of $\mathrm{Aut}(C)$ on $V := H^0(C, L)$ decomposes into irreducible representations, all of which have multiplicity zero or one.

Note that $\mathrm{Aut}(C)$ must be nontrivial. Hence, multiplicity free examples are very special in moduli.

In all the examples to follow, C will be 1-dimensional, and L will be $\omega_C^{\otimes \nu}$.

GIT basics

Let G act on V by a representation $G \to SL(V)$.

Kempf and Rousseau proved that when a point $x \in V$ is G-unstable, there is a "worst 1-PS" λ which is the most destabilizing.

Such a λ is not unique. It is preserved by a nontrivial parabolic $P_\lambda \subset\neq SL(V)$.

Kempf proved that $\mathrm{Stab}_G(x) \subseteq P_\lambda$.

Corollary (Kempf)

If the representation $\mathrm{Aut}(x) \to SL(V)$ *is irreducible, then* x *is stable.*

Proof: $\mathrm{Aut}(x)$ is not contained in any nontrivial parabolic $P_\lambda \subset\neq SL(V)$ $\square$.

Kempf applied this to prove GIT stability of Abelian varieties.

Unfortunately, there are very few smooth curves C and integers ν for which $(C, \omega^{\otimes \nu})$ is irreducible. So to use this strategy for moduli of curves, we need to generalize Kempf's result.

Proposition

Suppose (C, L) is multiplicity free. Let $\mathcal{B}$ be a basis of $H^0(C, L)$ adapted to the decomposition into irreducibles. Let T be the maximal torus acting diagonally with respect to $\mathcal{B}$. Then if C is m-Hilbert unstable, then there is a worst 1-PS in T.

Corollary

Let C, L, T be as above. Then if we compute $\mathrm{State}_m(C)$ and find that it contains the barycenter, then C is $SL(R+1)$-stable.

Proof for finite cylic groups

The proof in our preprint works for quite general groups.

In the special case that $\mathrm{Aut}(C)$ is a finite cyclic group, there is a simple proof:

We know $\mathrm{Aut}(C) \subset P_\lambda$. We want to show that when $\mathrm{Aut}(C)$ is diagonalized, the resulting diagonal torus $T \subset P_\lambda$.

We know $\mathrm{Aut}(C)$ can be simultaneously diagonalized by a matrix $\Psi \in SL(V)$. If we show that $\Psi \in P_\lambda$, then we are done.

Let A be a generator of $\rho(\mathrm{Aut}(C))$. For a finite cyclic group,

$$\text{multiplicity free} \iff A \text{ has distinct eigenvalues}$$

Then we can give a simple argument by finding and lifting eigenvectors.

- P can be put into block form, so that

$$M = \begin{pmatrix} M_1 & N_1 & * & \cdots & * \\ 0 & M_2 & * & \cdots & * \\ 0 & 0 & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & M_k \end{pmatrix},$$

  where each M_i is square of shape $n_i \times n_i$

- The characteristic polynomial of M is the product of the characteristic polynomials of the M_i. (The determinant of a block matrix like this is the product of the determinants, and $M - \lambda I$ is again a block matrix.)

  It follows that each M_i has distinct eigenvalues.

- *First block:* Let $\vec{w}_1, \ldots, \vec{w}_{n_1}$ be a set of eigenvectors for M_1 (so these are each of length n_1).

  These can be lifted to eigenvectors $\vec{v}_1, \ldots, \vec{v}_{n_1}$ of M by adding $n - n_1$ zeroes at the end of each $\vec{w}_i$.

- *Second block:* Let $\vec{w}_{n_1}, \ldots, \vec{w}_{n_1+n_2}$ be a set of eigenvectors for M_2, and write $\lambda_{n_1+1}, \ldots, \lambda_{n_1+n_2}$ for the corresponding eigenvalues. We lift these to eigenvectors $\vec{v}_j$ of M as follows: write

  $$\vec{v}_j = (v_j^{(1)}, \ldots, v_j^{(n_1)}, w_j^{(1)}, \ldots, w_j^{(n_2)}, 0, \ldots, 0)$$

  where the $v_j^{(\ell)}$ are to be determined.

We want $\vec{v}_j$ to be an eigenvector for λ_j:

$$(M_1 - \lambda_j \mathrm{Id}_{n_1}) \cdot \begin{pmatrix} v_j^{(1)} \\ \vdots \\ v_j^{(n_1)} \end{pmatrix} = -N_1 \cdot \begin{pmatrix} w_j^{(1)} \\ \vdots \\ w_j^{(n_2)} \end{pmatrix}$$

But the matrix $(M_1 - \lambda_j \mathrm{Id}_{n_1})$ has full rank, because λ_j is an eigenvalue of M_2, so by the assumption that the eigenvalues are distinct, λ_j is not an eigenvalue of M_1.

Thus we can lift eigenvectors to show that $\Psi \in P$.

In most GIT unstable examples, we are usually able to think of a destabilizing 1-PS quite easily using geometric intuition.

Open Question

Find an example where our intuition does not work.
That is: is there an example of the following kind?

- *G unstable*
- *not multiplicity free*
- *T stable for every "geometrically obvious" torus T*
- *but there is a worst 1-PS hiding in a torus which is not "geometrically obvious"*

For GIT unstable points, there is an interpretation of the worst 1-PS in terms of the state polytope.

Proposition

Suppose I is m-Hilbert unstable. Let T be a maximal torus containing a worst 1-PS. Then the worst 1-PS is the proximum (closest point) of $\mathrm{State}_{T,m}(I)$ *to the barycenter.*

Even for very old examples that we have known for a long time are GIT unstable, we can now ask: is the destabilizing 1-PS that we found the worst 1-PS?

Example: bicanonical Wiman curves

DEFINITION: we call the genus g hyperelliptic curves $y^2 = x^{2g+1} - 1$ the *Wiman curve* of genus g and denote it $\mathcal{W}_g$.

$\mathrm{Aut}(\mathcal{W}_g)$ is cyclic of order $4g + 2$.

To apply our strategy, we need to compute the state polytopes and check multiplicity-freeness for $H^0(C, 2K)$.

To compute the state polytope, we need equations for the bicanonically embedded curves.

In genus 3, $\deg K = 4 \Rightarrow \deg 2K = 8$
$h^0(2K) = 8 - 3 + 1 = 6$.

- We know $\mathcal{W}_3$ is a double cover of a $\mathbf{P}^1$.
- Embed $\mathbf{P}^1 \hookrightarrow \mathbf{P}^4$ as a rational normal curve of degree 4.

$$\begin{array}{ccccccccccc} & & a & : & b & : & c & : & d & : & e \\ [s,t] & \mapsto & [s^4 & : & s^3t & : & s^2t^2 & : & st^3 & : & t^4] \end{array}$$

Equations of the RNC are given by the 2×2-minors of

$$\begin{pmatrix} a & b & c & d \\ b & c & d & e \end{pmatrix}$$

$$I_{RNC} = (ac - b^2, ad - bc, ae - bd, bd - c^2, be - cd, ce - d^2)$$

- Add one more section y to get a map to $\mathbf{P}^5$: that is, identify $f \equiv y$. Then

$$I(\mathcal{W}_3) = I_{RNC} + (y^2 - x^7 + 1),$$

or

$$\begin{aligned} I(\mathcal{W}_3) &= (ac - b^2, ad - bc, ae - bd, \\ &\qquad bd - c^2, be - cd, ce - d^2, f^2 - ab + e^2) \end{aligned}$$

Aut(C) acts on (a, b, c, d, e, f) as $(\zeta_{14}^8, \zeta_{14}^6, \zeta_{14}^4, \zeta_{14}^2, 1, -1)$, so $(\mathcal{W}_3, 2K)$ is multiplicity free.
More generally, we can obtain equations for pluricanonical hyperelliptic curves by writing them as the intersection of a scroll with the quadrics of the form

$$x^k y^2 = x^k f(x).$$

And, for any g, $(\mathcal{W}_g, 2K)$ is multiplicity free.

We find that $I(\mathcal{W}_3)$ has 4615 initial ideals.

The highest regularity of these initial ideals is 19.
The maximum regularity for an ideal with Hilbert polynomial $p(t) = 8t - 2$ is 26.

$\mathcal{W}_3$ is unstable for $m = 2$
stable for $m \geq 3$.

This fits calculations by Hyeon and Lee: for $g = 3$, divisors of slope $28/3$ contract the hyperelliptic locus.

On the other hand, we can compute the polarization on the quotient: it is

$$[6\nu^2 m - 2\nu m - 2\nu + 1]\lambda - [\frac{1}{2}\nu^2 m]\delta$$

When $\nu = 2$ this is $[20m - 3]\lambda - [2m]\delta$, or slope $\frac{20m-3}{2m}$.
We solve $\frac{20m-3}{2m} = \frac{28}{3}$ to obtain $m = 9/4$.

For $m = 2$ the barycenter is $(7/3, \ldots, 7/3)$. We can compute the proximum:

$$\text{prox} = (\frac{12}{5}, \frac{12}{5}, \frac{12}{5}, \frac{12}{5}, \frac{12}{5}, \frac{10}{5})$$

This can be computed using the Maple package `Convex`. (We also checked the answer using the Kuhn–Tucker conditions.)

$$\text{prox} - \text{bary} = (1/15, \ldots, 1/15, -1/3).$$

Thus, the worst 1-PS is one which scales the span of the rational normal curve with equal weights and scales the cone with complementary weight.

For $g = 4, 5$ we cannot compute all the initial ideals of $\mathcal{W}_g$—there are too many.

But using a MONTE CARLO approach, we can compute enough initial ideals to see that the state polytope contains the barycenter for $m = 2, 3, 4, 5, 6$.

PREDICTION (Hyeon): hyperelliptic curves with $g \geq 4$ are stable for $m \geq 3/2$, unstable for $m < 3/2$.

Let w be the set of weights which scales the scroll variables with equal weights and scales the quadric variables with equal, complementary weight.
We confirmed using the `MUm` function that stability changes at $m = 3/2$ for this 1-PS.

For $g = 6, 7$ we have not been able to check stability even using our Monte Carlo approach.

Example: a ribbon R

DEFINITION: a ribbon on $\mathbf{P}^1$ is a double structure on $\mathbf{P}^1$—that is, a scheme R such that $R_{\text{red}} \cong \mathbf{P}^1$ and $\mathscr{I}^2_{R/R_{\text{red}}} = 0$.
Ribbons often arise as limits of canonical curves.

Bayer and Eisenbud give a genus four example in $\mathbf{P}^3$:
$R = (ac - b^2, ad^2 - 2bcd + c^3)$. This example has a $\mathbb{G}_m$-action with weights $-3, -1, 1, 3$.

R has twelve initial ideals $\{I_j\}$, all generated in degrees less than 6.

We can interpolate to find polynomials giving each $\text{OuterState}_m(I_j)$ and hence $\text{OuterState}_m(I)$ and $\text{State}_m(I)$ for any $m \geq 6$.

In general, we would expect the state polytope of an ideal in $\mathbf{P}^3$ would be three-dimensional. But $\text{State}_m(I)$ is only two-dimensional. The "extra" normal vector besides $(1,1,1,1)$ is $(-3,-1,1,3)$, the weight vector of the $\mathbb{G}_m$-action.

For example, when $m=6$, $\text{State}_m(I)$ is contained in the plane

$$\begin{aligned} a+b+c+d &= 306 \\ -3a-1b+c+3d &= -14. \end{aligned}$$

On the other hand, the barycenter does not satisfy the second equation. So the barycenter is outside the state polytope, since it is not even in the same plane.

The projection of the barycenter to this plane is inside $\text{State}_m(I)$, so the projection is the proximum.

We compute

$$\begin{aligned}
\texttt{MUm(I, -3,-1,1,3,6)} &= -56\\
\texttt{MUm(I, -3,-1,1,3,7)} &= -68\\
\texttt{MUm(I, -3,-1,1,3,8)} &= -80
\end{aligned}$$

$$\Longrightarrow \mu([C]_m, \rho) = -12m + 16.$$

We can also compute the Chow polytope of I.

It has four vertices.

This quadrilateral contains the barycenter of its ambient plane, so I is Chow semistable. Furthermore, we can see which initial ideals coalesce in the limit.

SUMMARY: This ribbon R is Hilbert unstable for all finite $m \geq 2$, but Chow strictly semistable.

Example: an elliptic bridge

Let X be a genus 5 nodal curve with an elliptic bridge:

$$X = \mathcal{W}_2 \cup E \cup \mathcal{W}_2,$$

where $\mathcal{W}_2$ is the hyperelliptic curve $y^2 = x^5 - 1$
and E is the elliptic curve $y^2 = x^3 - x$.

There are $500,094$ initial ideals, all generated in degrees ≤ 9.

We know X is Hilbert unstable for any finite m: we can use Hassett, Hyeon, and Lee's `MUm` function to find that the 1-PS with weights $2,2,2,2,2,1,0,2,2,2,2,2$ is destabilizing.

I have not been able to compute the proximum yet, so I do not know if this is the worst 1-PS.

However, we can interpolate and compute the Chow polytope. It has 1575 vertices and contains the barycenter of its ambient plane.

So, like the ribbon, this example is Hilbert unstable for any finite m, and Chow strictly semistable.

However, the picture for state polytopes is a little different — here, the state polytope is full-dimensional for every m, but the barycenter is outside it.