STUDY GUIDE FOR FINAL EXAM : VERSION 1.0
MATH 2500: SWINARSKI

The final exam will be cumulative. I am aiming for 40% material from Chapter 14, and the rest will be general.

On this study guide I'm only typing the Chapter 14 material (i.e. stuff that was new since Test 3). Look back at the old study guides as you study for the final.

Here's the usual disclaimer: The "main ideas" sections below may be incomplete. You are responsible for:

- things we covered in class and
- things that were on your homework assignments

even if I've accidentally forgotten to list them below.

If you find anything that looks incorrect below, please let me know!

1. Chapter 14

1.1. What we covered.

- We covered sections 14.1 through 14.8
- The only material I specifically remember leaving out was centers of mass of thin shells in Section 14.5

1.2. Main ideas on vector fields.

- A vector field is a map $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$ with m and n both ≥ 2. The two cases we've studied are two-dimensional vector fields $\mathbf{F}(x, y) = (P(x, y), Q(x, y))$ and three-dimensional vector fields $\mathbf{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$.
- There are three main questions we can try to answer about a vector field:
  1. Is $\mathbf{F}$ a gradient?
  2. What is the circulation of $\mathbf{F}$ along a path?
  3. What is the flux of $\mathbf{F}$ across a boundary?

1.3. Gradient vector fields.

- On a domain with no holes, the following are equivalent:
  1. $\mathbf{F}$ is a gradient
  2. $\mathbf{F}$ is conservative (the circulation around any closed loop is 0)
  3. $\mathbf{F}$ is path independent (if C_1 and C_2 are any two paths from a point p to a point q, then the circulations along C_1 and along C_2 are the same)
- To test whether $\mathbf{F}$ is a gradient:
  For a 2D field: Is $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$?
  For a 3D field: Is $\operatorname{curl} \mathbf{F} = \mathbf{0}$?
- The curl of a 3D field is given by

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

- If you determine that $\mathbf{F}$ is a gradient, and you want to find a potential function f (that is, $\mathbf{F} = \nabla f$), then you integrate and match.
- If $\mathbf{F} = \nabla f$, and C is a path from p to q, then

$$\int_C \mathbf{F} \cdot \mathbf{r}' = f(q) - f(p).$$

1.4. **2D vector fields: circulation and flux.**

- The circulation of a 2D vector field **F** along a curve C is given by
$$\text{2D circulation} = \int_a^b \mathbf{F}\cdot \mathbf{r}'(t)dt,$$
where $\mathbf{r}(t) = (x(t), y(t))$ is a parametrization of C and the domain of $\mathbf{r}$ is $a \leq t \leq b$.
- Here are two common physical interpretations of circulation integrals:
  - If **F** is a velocity field, then the circulation integral computes the amount of fluid flowing around C per unit time.
  - If **F** is a force field, then the circulation integral computes the work done by the field on a particle as it moves through the field.
- You might be given a curve C without being given the parametrization. You are responsible to know how to parametrize three kinds of curves: line segments, ellipses, and intersections of surfaces.
- To parametrize a line segment from a point p to a point q: let $\mathbf{v} = q-p$, and use $\mathbf{r}(t) = p+t\mathbf{v}$ with $0 \leq t \leq 1$.
- To parametrize an ellipse with center p and "arms" (semiaxes) **u** and **v**, use $\mathbf{r}(t) = p + \cos t\mathbf{u} + \sin t\mathbf{v}$ with $0 \leq t \leq 2\pi$. Note: this includes circles!
- The third kind of curve you might be asked to parametrize is a situation where you can set one of the variables equal to t and then solve to get the others. For instance: "Parametrize the intersection of $x = y^2$ and $z = y^3$." Well, if you set $y = t$, then you get $x = y^2 = t^2$ and $z = y^3 = t^3$, so $\mathbf{r}(t) = (t^2, t, t^3)$ will work.
- If C is a simple closed curve (i.e. it makes a loop and doesn't cross itself), then there is an alternative by Green's Theorem:
$$\text{2D circulation} = \oint_C \mathbf{F}\cdot\mathbf{r}' = \int\int_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$
where S is the region inside C. This version of Green's Theorem might turn a difficult line integral into an easy double integral, or a difficult double integral into an easy line integral. Note: here C is assumed to be counterclockwise.
- The flux of a 2D vector field **F** across a curve C is given by
$$\text{2D flux} = \int_a^b \mathbf{F}\cdot\left(\frac{dy}{dt}, -\frac{dx}{dt}\right) dt,$$
where $\mathbf{r}(t) = (x(t), y(t))$ is a parametrization of C and the domain of $\mathbf{r}$ is $a \leq t \leq b$.
- If C is a simple closed curve (i.e. it makes a loop and doesn't cross itself), then there is an alternative by Green's Theorem:
$$\text{2D flux} = \oint_C \mathbf{F}\cdot\left(\frac{dy}{dt}, -\frac{dx}{dt}\right) dt = \int\int_S \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$
where S is the region enclosed by C. Once again, we assume the parametrization on C is counterclockwise.

1.5. **3D vector fields: circulation and flux.**

- The circulation of a 3D vector field **F** along a curve C is given by
$$\text{3D circulation} = \int_a^b \mathbf{F}\cdot\mathbf{r}'(t)dt,$$
where $\mathbf{r}(t) = (x(t), y(t), z(t))$ is a parametrization of C and the domain of $\mathbf{r}$ is $a \leq t \leq b$. Note: this formula looks just like the formula for a 2D field, the only difference is that **F** and **r** have 3 components now.
- The boundary of a three dimensional region is a surface. So, the flux of a 3D vector fields is the flux across a surface. In contrast, the boundary of a two dimensional region in the x, y-plane is a curve. So the flux of 2D vector fields is the flux across a curve.

- The flux of a 3D vector field $\mathbf{F}$ across a surface S is given by
$$\text{3D flux definition} = \int\int_S \mathbf{F}\cdot\mathbf{n}d\sigma$$
But the version that is easier for computations is
$$\text{3D flux } = \int\int_D \mathbf{F}\cdot\left(\frac{\partial\mathbf{S}}{\partial u}\times\frac{\partial\mathbf{S}}{\partial v}\right)dudv$$
Here $\mathbf{S}(u,v)$ is a parametrization of the surface S, and D is the domain of $\mathbf{S}(u,v)$ in the u,v-plane.
- You are expected to know how to parametrize three kinds of surfaces: portions of spheres, portions of cylinders, and portions of explicit surfaces (remember these are surfaces of the form $z=f(x,y)$).
- The parametrization of a sphere of radius a comes from spherical coordinates with $\rho=a$, $\varphi=u$, and $\theta=v$.
$$\mathbf{S}(u,v) = (\overbrace{a\cos v\sin u}^{x=\rho\cos\theta\sin\varphi}, \overbrace{a\sin v\sin u}^{y=\rho\sin\theta\sin\varphi}, \overbrace{a\cos u}^{z=\rho\cos\varphi})$$
Then
$$\frac{\partial\mathbf{S}}{\partial u}\times\frac{\partial\mathbf{S}}{\partial v} = \underbrace{a^2\sin u}_{\rho^2\sin\varphi}\underbrace{(\cos v\sin u,\sin v\sin u,\cos u)}_{\text{sph. coords. with } \rho=1}$$
and
$$\left\|\frac{\partial\mathbf{S}}{\partial u}\times\frac{\partial\mathbf{S}}{\partial v}\right\| = a^2\sin u$$
If the surface is only a portion of a sphere, you need to figure out what bounds on u and v give that portion of a sphere.
- The parametrization of a cylinder of radius a and height h comes from cylindrical coordinates with $r=a$, $\theta=u$, and $z=v$.
$$\mathbf{S}(u,v) = (\overbrace{a\cos u}^{x=r\cos\theta}, \overbrace{a\sin u}^{y=r\sin\theta}, \overbrace{v}^{z})$$
Then
$$\frac{\partial\mathbf{S}}{\partial u}\times\frac{\partial\mathbf{S}}{\partial v} = \underbrace{a}_{r}\ \underbrace{(\cos u,\sin u,z)}_{\text{like a circle of radius 1}}$$
and
$$\left\|\frac{\partial\mathbf{S}}{\partial u}\times\frac{\partial\mathbf{S}}{\partial v}\right\| = a$$
(because $r=a$ and $dxdy$ is $rdrd\theta$ in polar coordinates.) If the surface is only a portion of a cylinder, you need to figure out what bounds on u and v give that portion of a cylinder.
- The last kind of surface you may be asked to parametrize is an explicit surface, that is, a surface of the form $z=f(x,y)$. Then set $u=x$ and $v=y$. Then the parametrization is
$$\mathbf{S}(u,v) = (u,v,f(u,v))$$
and you would need to compute $\frac{\partial\mathbf{S}}{\partial u}\times\frac{\partial\mathbf{S}}{\partial v}$ and $\|\frac{\partial\mathbf{S}}{\partial u}\times\frac{\partial\mathbf{S}}{\partial v}\|$.
- The divergence of a 3D vector field $\mathbf{F}=(P,Q,R)$ is $\operatorname{div}\mathbf{F} = \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}$.
- If S is a closed surface (i.e. it encloses a 3D region $\mathscr{R}$) then the Divergence Theorem gives an alternative way to compute flux:
$$\text{3D flux} = \int\int_S \mathbf{F}\cdot\mathbf{n}d\sigma = \int\int\int_{\mathscr{R}} \operatorname{div}\mathbf{F}dV$$
Notice that if you use the Divergence Theorem you never have to parametrize the surface.

- If S is a surface and C is its boundary, then Stokes's Theorem gives an alternative way to compute the circulation:
$$\int_C \mathbf{F} \cdot \mathbf{r}' = \int\int_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n} d\sigma$$
- Note the difference between the setups of the Divergence Theorem and Stokes's Theorem. The Divergence Theorem is for a closed surface (which has no boundary curve). Stokes's Theorem is for a surface with a boundary curve (so it does not enclose a region).
- We may never get to this in class, but you can check: $\operatorname{div} \operatorname{curl} \mathbf{F} = 0$.

1.6. Miscellaneous.

- Memorize
$$\int \sin^2 x dx = \frac{x}{2} - \frac{1}{4} \sin(2x) + C$$
$$\int \cos^2 x dx = \frac{x}{2} + \frac{1}{4} \sin(2x) + C$$
- The arclength of a parametrized curve between $p = \mathbf{r}(a)$ and $q = \mathbf{r}(b)$ is
$$\text{arclength} = \int_a^b \|\mathbf{r}'(t)\| dt$$
- The surface area of a parametrized surface is
$$\text{surface area} = \int\int_D \left\| \frac{\partial \mathbf{S}}{\partial u} \times \frac{\partial \mathbf{S}}{\partial v} \right\| dudv$$
where D is the domain of the parametrization $\mathbf{S}(u, v)$.