MATH 2500 - Swinarski
Midterm 3
April 15, 2009

No calculators, computers, or other electronic devices are permitted.
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No books, notes, or formula sheets are permitted.
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Make sure to show all work.
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You have 50 minutes to complete this exam.

Name : Solutions

Good luck!

Question	Points	Score
1	12	
2	12	
3	12	
4	4	
5	12	
6	24	
7	12	
8	12	
Total:	100	

1. (12 points) Evaluate the following double integral.

$$\int_0^1 \int_0^{2-x} (x^2 - y)dydx$$

$$= \int_{x=0}^{1} \left[x^2 y - \frac{y^2}{2} \Big|_0^{2-x} \right] dx$$

$$= \int_{x=0}^{1} \left[x^2(2-x) - \frac{(2-x)^2}{2} \right] dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \left[4x^2 - 2x^3 - x^2 + 4x - 4 \right] dx$$

$$= \frac{1}{2} \int_{x=0}^{1} \left[-2x^3 + 3x^2 + 4x - 4 \right] dx$$

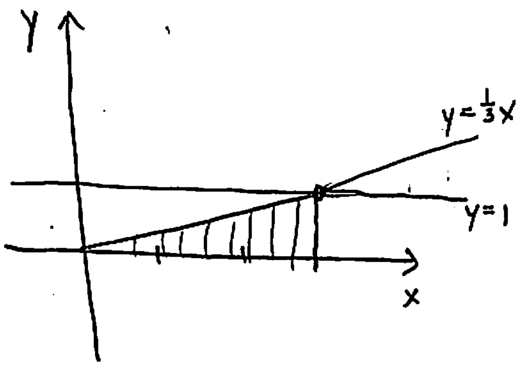
$$= \frac{1}{2} \left(-\frac{1}{2}x^4 + x^3 + 2x^2 - 4x \Big|_0^1 \right)$$

$$= \frac{1}{2} \left(-\frac{1}{2} + 1 + 2 - 4 \right)$$

$$= \frac{1}{2} \left(-\frac{3}{2} \right) = -\frac{3}{4}$$

2. (12 points) Draw a picture of the region in the x, y-plane described by the bounds of the following integral. Change the order of integration, and evaluate the integral.

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx dy$$



$x = 3y$, or $y = \frac{1}{3}x$

$$= \int_{x=0}^{3} \int_{y=0}^{\frac{1}{3}x} e^{x^2}\, dy\, dx$$

$$= \frac{1}{3} \int_{x=0}^{3} x e^{x^2}\, dx$$

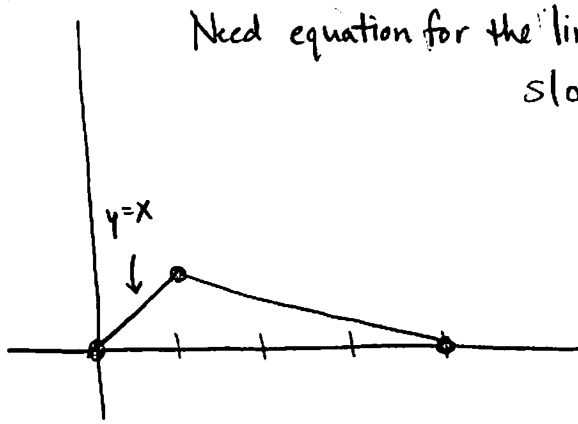
Let $u = x^2$ $\quad x = 3 \Rightarrow u = 9$

$du = 2x dx$ $\quad x = 0 \Rightarrow u = 0$

$$= \frac{1}{6} \int_{u=0}^{9} e^{u}\, du$$

$$= \frac{1}{6} (e^9 - 1)$$

3. (12 points) Suppose T is a thin triangular plate with vertices $(0,0)$, $(1,1)$ and $(4,0)$. Suppose that the density of T is given by the function $\delta(x,y) = x$. Set up all the integrals needed to find the center of mass of T. You do not need to evaluate these integrals.



Need equation for the line through $(1,1)$ and $(4,0)$:

$$\text{slope} = \frac{0-1}{4-1} = -\frac{1}{3}$$

$$y - 0 = -\frac{1}{3}(x-4)$$

$$y = -\frac{1}{3}x + \frac{4}{3}$$

or $$-\frac{1}{3}x = y - \frac{4}{3}$$

$$x = -3y + 4$$

$$M = \int_{y=0}^{1} \int_{x=y}^{-3y+4} \overset{\delta}{x} \, dx\, dy$$

$$M_y = \int_{y=0}^{1} \int_{x=y}^{-3y+4} x^2 \, dx\, dy$$

$$M_x = \int_{y=0}^{1} \int_{x=y}^{-3y+4} xy \, dx\, dy$$

$$\vec{x} = \frac{M_y}{M} \qquad \bar{y} = \frac{M_x}{M}$$

4. (4 points) Please supply the following formulas.

To switch between cylindrical and rectangular coordinates, one uses the formulas

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

and

$$dxdydz = r\, dr\, dz\, d\theta$$

To switch between spherical and rectangular coordinates, one uses the formulas

$$\begin{aligned} x &= \rho\cos\theta\sin\varphi \\ y &= \rho\sin\theta\sin\varphi \\ z &= \rho\cos\varphi \end{aligned}$$

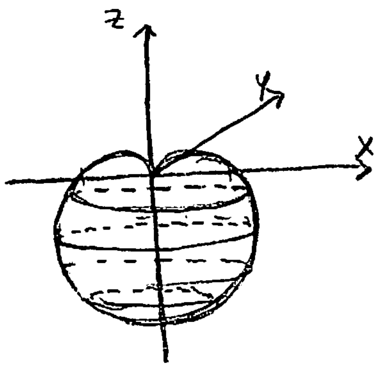
and

$$dxdydz = \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

5. (12 points) Let P be the peach given by the equation

$$x^2 + y^2 + z^2 = \sqrt{x^2 + y^2 + z^2} \quad - \quad z$$

in rectangular coordinates. P is pictured below. Set up an integral in spherical coordinates to compute the volume of P. You do not need to evaluate it yet.



Switch the equation above to spherical:

$$\rho^2 = \rho - \rho\cos\varphi$$

$$\Rightarrow \quad \rho = 1 - \cos\varphi$$

Note: no θ — the peach has circular symmetry around the z-axis

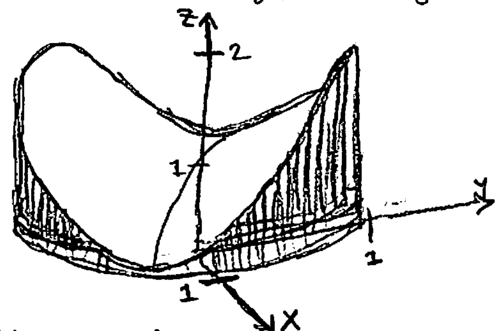
Also from the picture we see φ must go all the way from 0 to π

And the origin is in the peach, so ρ starts at 0

Volume of peach

$$= \int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} \int_{\rho=0}^{1-\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

6. Let R be the region bounded below by the x,y-plane, on its sides by the cylinder $x^2+y^2=1$, and above by the saddle-like surface $z=xy+1$. The region R is pictured below.



(a) (12 points) Set up an integral in rectangular coordinates to compute the volume of R. You do not need to evaluate it yet.

Projection to xy plane is the standard unit circle

$$\text{Volume} = \int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=0}^{xy+1} 1 \, dz\, dy\, dx$$

(b) (12 points) Set up an integral in cylindrical coordinates to compute the volume of R. You do not need to evaluate it yet.

Change $z = xy+1$ to cylindrical:

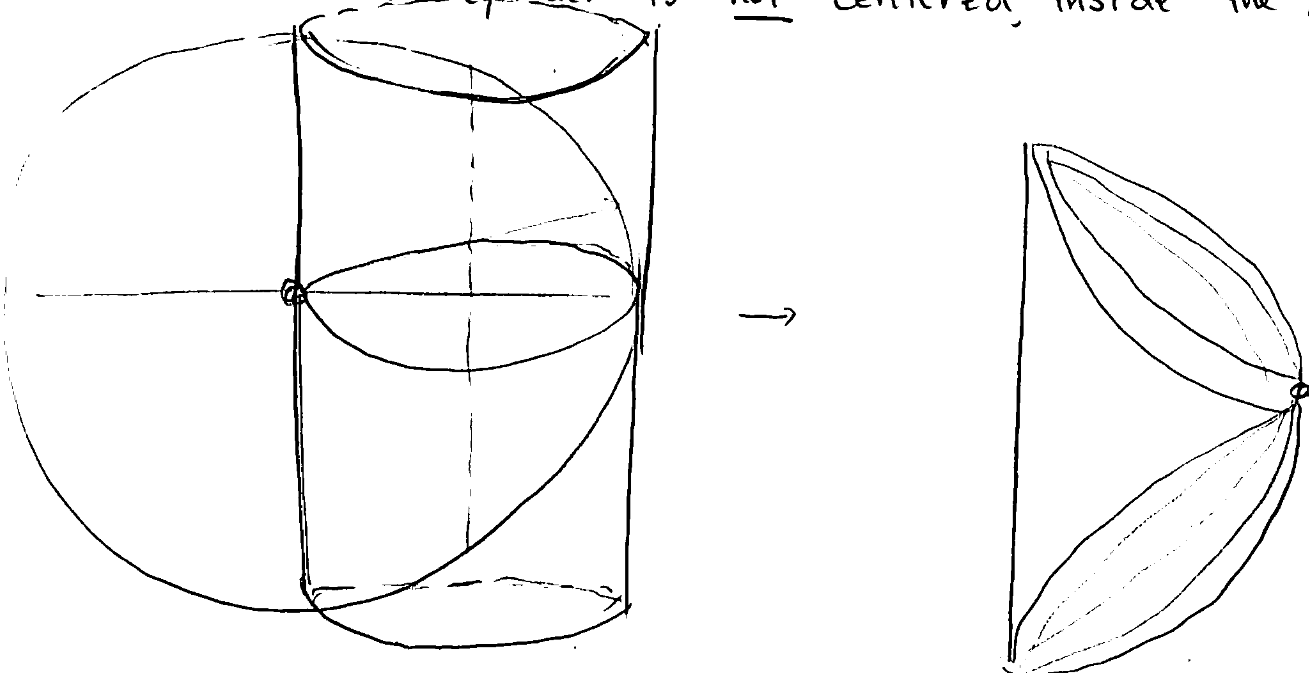
$$z = r^2 \cos\theta \sin\theta + 1$$

$$= \frac{1}{2} r^2 \sin(2\theta) + 1$$

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{\frac{1}{2}r^2\sin(2\theta)+1} r \, dz\, dr\, d\theta$$

7. (12 points) Let S be a sphere of radius 2. Let C be a cylinder whose cross sectional circle has radius 1, and whose side lies along a diameter of the sphere. Let I be the intersection of S and C. Draw a picture of I, and set up an integral to compute the volume of ~~S~~ I. You may use any coordinate system you like. You do not need to evaluate the integral yet.

Note: the cylinder is not centered inside the sphere



Let's set this up in cylindrical

I used the sphere $x^2+y^2+z^2=4 \rightarrow r^2+z^2=4$

and the cylinder $(x-1)^2+y^2=1 \longrightarrow r=2\cos\theta$

Then

$$\text{volume } I = \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \int_{r=0}^{2\cos\theta} \int_{z=-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$$

8. (12 points) Choose any one of the four integrals that you set up in Question 5, 6, or 7, and evaluate it.

From #5: $\int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi} \int_{\rho=0}^{1-\cos\varphi} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

$\frac{\rho^3}{3} \sin\varphi \Big|_0^{1-\cos\varphi}$

$$= \int_{\theta=0}^{2\pi} \cdot \int_{\varphi=0}^{\pi} \frac{1}{3} \sin\varphi \, (1-\cos\varphi)^3 \, d\varphi \, d\theta$$

Let $u = (1-\cos\varphi)$

$du = \sin\varphi \, d\varphi$

$$= \frac{1}{3} \int_{\theta=0}^{2\pi} \left[\frac{(1-\cos\varphi)^4}{4} \Big|_0^{\pi} \right] d\theta$$

$$= \frac{1}{3} \int_{\theta=0}^{2\pi} \left[\frac{2^4}{4} - \frac{1^4}{4} \right] d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} 4 \, d\theta = \frac{8\pi}{3}.$$

cont. →

From #6a: $$\int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{z=0}^{xy+1} 1 \; dz\, dy\, dx$$

$$= \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (xy+1)\; dy\; dx$$

$$= \int_{x=-1}^{1} \left[x\frac{y^2}{2} + y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \right] dx$$

$$= \int_{x=-1}^{1} 2\sqrt{1-x^2}\; dx$$

$$= 2 \int_{x=-1}^{1} \sqrt{1-x^2}\; dx \quad \leftarrow \quad \text{area of semicircle of radius 1}$$

$$= 2\cdot\left(\frac{\pi}{2}\right) = \pi$$

From #6b: $$\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{\frac{1}{2}r^2\sin(2\theta)+1} r\; dz\; dr\; d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\left(\frac{1}{2}r^3\sin(2\theta) + r\right) dr\; d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[\frac{1}{2}\frac{r^4}{4}\sin 2\theta + \frac{r^2}{2}\Big|_0^1\right] d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[\frac{1}{8}\sin 2\theta + \frac{1}{2}\right] d\theta$$

$$= \frac{1}{8}\int_{\theta=0}^{2\pi}\sin(2\theta)\,d\theta + \frac{1}{2}\int_{\theta=0}^{2\pi} d\theta$$

$$= \frac{1}{8}\cdot 0 + \frac{1}{2}\cdot 2\pi = \pi$$

From #7: $\displaystyle\int_{\theta=-\frac{\pi}{2}}^{\pi/2} \int_{r=0}^{2\cos\theta} \int_{z=-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$

$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{2\cos\theta} 2r\sqrt{4-r^2}\,dr\,d\theta$$

Let $u = 4-r^2$ $\qquad r=2\cos\theta \Rightarrow u = 4-4\cos^2\theta$

$du = -2r\,dr$ $\qquad = 4\sin^2\theta$

$r=0 \Rightarrow u=4$

$$= \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_{u=4}^{4\sin^2\theta} -\sqrt{u}\,du \right] d\theta$$

$$-\frac{2}{3}u^{3/2}\Big|_{4}^{4\sin^2\theta}$$

Be careful! $(4\sin^2\theta)$ is positive no matter whether $\sin\theta$ is $+$ or $-$

So $(4\sin^2\theta)^{3/2} \neq 8\sin^3\theta$

Instead $(4\sin^2\theta)^{3/2} = |8\sin^3\theta|$

$$= \int_{\theta=-\frac{\pi}{2}}^{\pi/2} \frac{16}{3} - \frac{2}{3}|8\sin^3\theta|\,d\theta$$

$$= \underbrace{\int_{\theta=-\frac{\pi}{2}}^{\pi/2} \frac{16}{3}\,d\theta}_{\frac{16\pi}{3}} - \frac{16}{3}\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin^3\theta|\,d\theta$$

What is $\displaystyle\int_{\theta=-\frac{\pi}{2}}^{\pi/2} |\sin^3\theta|\,d\theta$?

$$= \int_{\theta=-\frac{\pi}{2}}^{0} -(\sin^3\theta)\,d\theta + \int_{0}^{\pi/2} \sin^3\theta\,d\theta$$

Cont. →

$$\int_0^{\pi/2} \sin^3\theta \, d\theta = \int_0^{\pi/2} \sin\theta \left(1-\cos^2\theta\right) d\theta$$

$$= -\cos\theta + \frac{\cos^3\theta}{3} \Big|_0^{\pi/2}$$

$$= 0 - \left(-1 + \frac{1}{3}\right) = \frac{2}{3}$$

Similarly, find

$$\int_{-\frac{\pi}{2}}^{0} -\sin^3\theta \, d\theta = \frac{2}{3}$$

So

$$\text{Volume of } I = \frac{16\pi}{3} - \frac{16}{3}\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin^3\theta| \, d\theta$$

$$= \frac{16\pi}{3} - \frac{16}{3}\cdot\frac{4}{3}$$

$$= \frac{16\pi}{3} - \frac{64}{9}$$

Note: we could have avoided the issue of $|\sin^3\theta|$ if we had used the fact that I is symmetric across the x,z-plane, and just worked in the "quadrant" where $\sin\theta$ is positive