RESEARCH STATEMENT

EMILLE K. DAVIE

My research is in the area of mapping class groups of surfaces with a focus on the braid group. In particular, I have considered several questions about the monoid of right-veering diffeomorphisms of a surface with nonempty boundary. These right-veering diffeomorphisms play a large role in contact topology by categorizing precisely when a contact structure on a 3-manifold is tight. I am working to classify right-veering maps of a genus one surface with one boundary component by considering its image under the Burau representation of the 3-strand braid group, B_3. This representation is well-studied and known to be faithful for B_3. I have found necessary and sufficient conditions for a map h to be right-veering in the case that h is reducible or periodic.

1. BACKGROUND

Throughout, let Σ denote a compact, orientable surface with nonempty boundary. The (restricted) *mapping class group* of Σ, denoted $\mathrm{Mod}(\Sigma, \partial\Sigma)$, is the group of isotopy classes of orientation preserving diffeomorphisms of Σ which restrict to the identity on $\partial\Sigma$. Let α and β be isotopy classes of properly embedded oriented arcs in Σ based at $x \in \partial\Sigma$ which intersect minimally. Consider the vectors v_α and v_β which are tangent to α and β, respectively, at x. We say that β is *to the right* of α if $\{v_\beta, v_\alpha\}$ corresponds to the orientation of Σ at x. We define the monoid of right-veering maps.

Definition 1. *Let h represent an element of $\mathrm{Mod}(\Sigma, \partial\Sigma)$. Then h is right-veering if for every $x \in \partial\Sigma$ and every properly embedded oriented arc α based at x, we have that $h(\alpha)$ is to the right of α. We denote the set of all such maps $\mathrm{Veer}(\Sigma, \partial\Sigma) = \mathrm{Veer}$.*

These diffeomorphisms are especially helpful in contact topology. An *open book decomposition* (Σ, h) for a 3-manifold M with binding K is a homeomorphism between $((\Sigma \times [0,1])/_{\sim_h}, (\partial\Sigma \times [0,1])/_{\sim_h})$ and (M, K). The equivalence relation $\sim_h$ is generated by $(x, 0) \sim_h (h(x), 1)$ for $x \in \Sigma$ and $(y, t) \sim_h (y, t')$ for $y \in \partial\Sigma$. In [1] Honda, Kazez, Matic prove that a contact 3-manifold (M, ξ) is tight if and only if all of its open book decompositions (Σ, h) have $h \in \mathrm{Veer}$. Furthermore, they prove in [2] that when Σ is a genus one surface with one boundary component, it suffices that just one open book have right-veering monodromy.

Date: December 2006.

For S, a once punctured torus, let C_1 and C_2 be dual closed curves in S and let T_i be a right Dehn twist about C_i. The map $B_3 \to \mathrm{Mod}(S, \partial S)$ which sends generator σ_i to twist T_i is an isomorphism. This isomorphism provides a way to conveniently view elements of B_3 as diffeomorphisms of S. Using this fact, we are able to find useful results about right-veering maps of S using the Burau representation of the braid group $\rho : B_n \to GL_{n-1}(\mathbb{Z}[t, t^{-1}])$ which is well known to be faithful for $n = 3$. We proceed here by appealing to the Thurston classification for surface diffeomorphisms, which states that every representative f in the mapping class group of a compact, orientable surface without boundary is either reducible, periodic, or pseudo-Anosov.

2. REDUCIBLE MAPPING CLASSES

If we let T^2 denote the torus, there is a natural surjection $\delta : \mathrm{Mod}(S, \partial S) \to \mathrm{Mod}(T^2) \cong SL_2\mathbb{Z}$, and ker δ is generated by the element Δ^4. Here Δ denotes the Garside braid, $\sigma_1 \sigma_2 \sigma_1$. Geometrically, this is a positive half-twist of all the strands. In a slight abuse of terminology, we say that an element $\omega \in B_3$ is *reducible (periodic, pseudo-Anosov)* if its image $\widehat{\omega}$ in $SL_2\mathbb{Z}$ is reducible (periodic, pseudo-Anosov). Reducible elements of $\mathrm{Mod}(S, \partial S)$ leave some essential simple closed curve invariant, and therefore in the braid group, it is a conjugate of some power of σ_1 plus possibly some boundary twisting. More precisely, we have the following proposition.

Proposition 1. *An element $\omega \in B_3$ is reducible if and only if it can be written in the form $\omega = \Delta^{2k} \tau\, {\sigma_1}^m\, \tau^{-1}$ for some $\tau \in B_3$ and $k, m \in \mathbb{Z}$.*

Honda, Kazez, Matic prove in [2] that a reducible map $\omega = \Delta^{2k} \tau\, {\sigma_1}^m\, \tau^{-1}$ is right-veering if and only if $k > 0$ or $k = 0$ and $m \geq 0$. We extract k and m by the following. For a braid $\beta \in B_3$, let $\overline{\beta}$ denote its image in $GL_2(\mathbb{Z}[t, t^{-1}])$ under the Burau representation. Since $\overline{\Delta^{2n}} = \begin{pmatrix} t^{3n} & 0 \\ 0 & t^{3n} \end{pmatrix}$ and since trace is conjugation invariant, it can be verified that $\mathrm{trace}(\overline{\omega}) = t^{3k} + (-1)^m t^{3k+m}$. Several immediate observations can be made from this nice form of the trace, but it is not enough to determine k and m completely. For example, if $\mathrm{trace}(\overline{\omega}) = t^6 + t^{-6}$, it is impossible to determine if $k = 2$ and $m = -12$ or if $k = -2$ and $m = 12$. Moreover, this distinction is necessary since the former implies ω is right-veering and the latter that ω is left-veering. The following theorem gives us sufficient information to determine k and m assuming $\mathrm{trace}(\overline{\omega})$ and the eigenvalue λ for $\widehat{\omega}$ are known.

Theorem 1. *Suppose that ω is a nontrivial reducible map. Then $|m|$ is the greatest common divisor of the skew diagonal of $\widehat{\omega}$. Moreover, for $\lambda = 1$, m takes the sign of the upper entry in the skew diagonal, and for $\lambda = -1$, m takes the sign of the lower entry in the skew diagonal.*

3. PERIODIC, IRREDUCIBLE MAPPING CLASSES

Up to conjugation, there are only six periodic elements of $SL_2\mathbb{Z}$. These elements are $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and their negatives. It follows that the periodic, irreducible elements of B_3 are conjugates of $(\sigma_1\,\sigma_2)^i$, where i is an integer not divisible by 3 (else the map is reducible) and conjugates of Δ^{2r+1} for $r \in \mathbb{Z}$ (even powers of Δ are reducible). We have the following theorem.

Theorem 2. *Let ω be a nontrivial periodic map. Then $\overline{\omega^{12}} = \begin{pmatrix} t^{6j} & 0 \\ 0 & t^{6j} \end{pmatrix}$ for some integer j. Moreover, ω^{12} is right-veering if and only if $j > 0$.*

The idea here is that if ω is periodic, ω^{12} must be in ker δ. That is, ω^{12} is a power of a Dehn twist about the boundary of S. Thus the sign of j tells us if this twist is a positive or a negative one. Now, the following lemma and corollary gives us the desired information about ω.

Lemma 1. *h^n is right-veering if and only if h is right-veering.*

Corollary 1. *A nontrivial periodic map ω is right-veering if and only if $\overline{\omega^{12}}$ has trace $2t^k$ for some positive integer k.*

4. FUTURE RESEARCH

My plans for the immediate future include continuing my ongoing work to characterize pseudo-Anosov right-veering maps. This case will reveal much about the geometry involved in the Burau representation. In [2], the authors establish a relationship between the linking number (sometimes referred to as the exponent sum) of a braid ω, the rotation number of ω, and the value of the Rademacher function on $\overline{\omega}$. They give a sufficient condition for a map to be right-veering based upon its rotation number. The linking number shows up as the power of t in the determinant of the image of ω under the Burau representation. I also believe that the rotation number can be extracted from this matrix, as well.

I am also interested in the Lawrence-Krammer representation of B_n which is known to be faithful for all n. Thus characterizing right-veering 3-strand braids via this representation could be a natural next step. In general, I am curious about the existence of faithful finite dimensional representations for the mapping class group $\mathrm{Mod}(\Sigma_{g,b,n})$, where $\Sigma_{g,b,n}$ is a surface of genus g with b boundary components and n punctures. There are faithful representations for several classes of triples (g, b, n), such as $(0, 0, n)$ and $(0, 1, n)$, but this problem is unsolved for general (g, b, n).

I also plan to continue my ongoing research with Ken Baker, a postdoc at the Georgia Institute of Technology, involving braiding in lens spaces. This collaboration started as a project in a VIGRE group. We have formulated in [3] an analogue for Alexander's theorem for knots in S^3 to knots in arbitrary lens spaces. More specifically, we have the following result for a knot K in a lens space L.

Theorem 3. *Every knot in a lens space may be braided.*

Here, we say a knot may be braided in L if it is isotopic to a knot which is transverse to each meridional disk in one of the Heegaard solid tori. We show that K may be braided by performing a finite sequence of moves across compressing disks in L.

REFERENCES

[1] K. Honda, W. Kazez and G. Matić, *Right-veering diffeomorphisms of compact surfaces with boundary I*, preprint 2005. `ArXiv:math.GT/0510639.`

[2] K. Honda, W. Kazez and G. Matić, *Right-veering diffeomorphisms of compact surfaces with boundary II*, preprint 2006. `ArXiv:math.GT/0603626.`

[3] *Braiding in lens spaces*, (with K. Baker, V. Hower, and A. Johnson), preprint in preparation.

UNIVERSITY OF GEORGIA, ATHENS, GEORGIA 30602
E-mail address: `davie@math.uga.edu`
URL: `www.math.uga.edu/~davie`