(5) A company would like to construct a cylindrical can with an open top with a volume of 4π cubic feet. If the cost of the materials for the sides of the can is \$2 per square foot, and the cost of materials for the bottom of the can is \$5 per square foot, what are the dimensions of the can (radius, height) which will minimize the cost of the materials?

Potentially useful formulas:

Volume of a cylinder $= \pi r^2 h$ Surface area around a cylinder $= 2\pi r h$

Area of a circle $= \pi r^2$

Find the dimensions of the can (radius, height) which achieve the minimum cost of materials.

Solution. In this problem we are trying to minimize the cost of the materials. We can find a formula for this cost as:

$$\text{(cost)} = \text{(cost of sides)} + \text{(cost of bottom)}$$

The cost of the sides is 2(area of sides) and the cost of the bottom of 5(area of bottom). In particular, we have (using the formulas given):

$$C = 5\pi r^2 + 2 \cdot 2\pi r h = 5\pi r^2 + 4\pi r h$$

where r is the radius of the can, h is the height, and C is the total cost. Now, we need to eliminate one of these variables in order to get just one variable. To do this, we use the fact that the volume is fixed at 4π cubic feet. Using the formula above this says:

$$4\pi = \pi r^2 h$$

We can then solve this for h to get:

$$h = \frac{4}{r^2}$$

Substituting this into the equation for C, we obtain C as a function only of r:

$$C = C(r) = 5\pi r^2 + 4\pi r \left(\frac{4}{r^2}\right) = 5\pi r^2 + 16\pi/r$$

Now we need to determine the interval on which this function is defined — that is to say, we need to find which values of r which satisfy the constraints of the problem. In this case, or course, r certainly cannot be negative. Besides this, the only constraint we actually have is that the volume of the can must be equal to 4. This means that it doesn't make sense to have the radius equal to 0 (since such a can would have no volume). On the other hand, we can make the radius very small, since by then making the can very tall, we would still be able to have the required volume of 4. This says so far:

$$0 < r$$

Now we need to ask how large r could be. If we made r very large, so that the base is very wide, we ask whether we might still be able to ensure that the volume is 4. We can do this by making the can sufficiently short. Therefore we can make the radius as big as we like. This says that the interval is:

$$(0, \infty)$$

Therefore we cannot use the closed interval method, but instead must use the first derivative test (i.e. make a sign chart for the derivative).

The next step is to find the critical points for $C(r)$. We first take the derivative:

$$C'(r) = 5\pi \cdot 2r + 16\pi(-1)r^{-2} = 10\pi r - 16\pi/r^2$$

The only place this is not defined would be at $r = 0$, but this is not in the interval, so we don't think of this as a critical point. We therefore consider where $C'(r) = 0$:

$$0 = C'(r) = 10\pi r - 16\pi/r^2$$

gives:

$$16\pi/r^2 = 10\pi r$$

multiplying both sides by r^2 and dividing both sides by 10π:

$$16/10 = r^3$$

or

$$8/5 = r^3$$

so $r = \sqrt[3]{8/5}$ (which is the same as $2/\sqrt[3]{5}$).

To check whether or not this is a max or a min (we hope it is a min), we construct a sign chart for the first derivative.

I can't really draw this (since I don't have a good drawing program or scanner), so I'll try to talk through the chart.

The sign chart would be drawn on a number line and we only need to worry about the part starting from (but not including) 0 and continuing all the way to the right. We only need to mark the single critical point $r = \sqrt[3]{8/5}$. Now we just need to test whether $C'(r)$ is positive or negative between 0 and $\sqrt[3]{8/5}$ and then whether or not it is positive or negative to the right or $\sqrt[3]{8/5}$.

Let's plug in some values now :

$$C'(1) = 10\pi 1 - 16\pi/1^2 = 10\pi - 16\pi = -6\pi$$

is negative and

$$C'(2) = 10\pi 2 - 16\pi/2^2 = 20\pi - 16/4\pi = 20\pi - 4\pi = 16\pi$$

is positive. Therefore $C'(r)$ is negative somewhere, but then is positive to the right of that. This means the sign chart must change from negative to the left of $\sqrt[3]{8/5}$ to positive on the right of $\sqrt[3]{8/5}$. Therefore the function $C(r)$ is decreasing on the left and increasing on the right. The first derivative test says that therefore $C(r)$ has a minimum at $r = \sqrt[3]{8/5}$.

Therefore to build the cheapest can, we must make $r = \sqrt[3]{8/5}$. As for the height, we can solve the equation from before:

$$h = \frac{4}{r^2}$$

so

$$h = \frac{4}{(\sqrt[3]{8/5})^2} = \frac{4}{(8/5)^{2/3}} = 4(8/5)^{-2/3}$$

This can be simplified slightly, but I'm not going to worry about it.

□

(6) A farmer would like to set up a rectangular enclosure of fencing in order to enclose a fixed area of 1000 square feet. If three of the sides of the enclosure cost \$2 per linear foot of fencing, and the fourth side costs \$7 per linear foot, find the dimensions of the enclosure which minimize the total cost.

Find the dimensions of the fence (length, width, height) which minimize the total cost.

Solution. The quantity we are trying to minimize is the cost. We can write the equation for this as:

$$\text{(cost)} = \text{(cost of first 3 sides)} + \text{(cost of 4th side)}$$

The cost of the first 3 sides is the total length of the first 3 sides times 2. The cost of the 4'th side is the length of this side times 7. This gives: supposing the enclosure has width w and length l, then we can write

$$C = 2(l + w + l) + 7w = 4l + 8w$$

(of course which you call length and width depends on how you draw it).

Now we have too many variables, and we want to get rid of one. To do this we use the fact that the area is fixed at 1000, so we have:

$$1000 = lw$$

So, for example, $l = 1000/w$. Plugging into the equation for cost, we get:

$$C = C(w) = 4(1000/w) + 8w = 8w + 4000/w$$

We now need to find the interval on which this is defined. The logic here is essentially identical to the logic on the previous problem. We can make the width as small as we want and keep the total area at 1000 by making the enclosure longer and longer. This says that we can get w as close as we want to 0, but not actually 0 (since then the enclosure would have no area, and we need the area to always be 1000). On the other hand, we can make w as large as we want without contradicting our requirement that the total area is 1000 by making the length small. Therefore we have the interval

$$(0, \infty)$$

Now for critical points. We find first the derivative:

$$C'(w) = 8 + 4000(-1/w^2) = 8 - 4000/w^2$$

This is only undefined at $w = 0$, which is not in our interval. Therefore we need only consider $C'(w) = 0$. This gives

$$8 = 4000/w^2$$

and re-arranging gives:

$$w^2 = 4000/8 = 1000/2 = 500$$

so $w = \sqrt{500} = \sqrt{5} \cdot 100 = 10\sqrt{5}$.

As before, we need now to construct a sign chart for the derivative (we can't use the closed interval method, since our interval is not closed). Our sign chart will take place on the right half of the number line, to the right of 0, and with the single point $10\sqrt{5}$ drawn. We need to check whether $C'(w)$ is positive or negative to the left of $10\sqrt{5}$, and then whether it is positive or negative to the right of $10\sqrt{5}$.

Let's check some values. Plugging in $w = 1$, we get:

$$C'(1) = 8 - 4000/1^2 = 8 - 4000 = \text{negative}$$

and

$$C'(10000) = 8 - 4000/(10000)^2 = 8 - 4000/(100000000) = \text{positive}$$

therefore we find $C'(w)$ is negative on the left and positive on the right of $10\sqrt{5}$. In particular, this means that $C(w)$ is decreasing to the left of $10\sqrt{5}$ and increasing to the right of $10\sqrt{5}$. The first derivative test therefore tells us that $C(w)$ therefore has a minimum when $w = 10\sqrt{5}$

Now lets figure out what the length is. We use $l = 1000/w$ from above and find:

$$l = 1000/(10\sqrt{5}) = 100/\sqrt{5} = 100\sqrt{5}/5 = 20\sqrt{5}$$

So the desired dimensions are $10/sqrt5$ by $20\sqrt{5}$. □

(7) A company would like to construct a box with a square base in such as way as to enclose the maximum possible volume at a fixed cost of \$100. The sides and the bottom of this box will each cost \$2 per square foot, and the top will cost \$7 per square foot. Find the dimensions of the box which will maximize the volume at this cost.

Find the dimensions of the box (length, width, height) which achieve the maximum volume at the price of 100.

Solution. In this problem we want to maximize the volume. We may write an equation for volume as:

$$V = lwh$$

where l is the length, w the width and h the height. Now, since the box has a square base, we immediately have that $l = w$. So really teh equation is:

$$V = w^2 h$$

We still have too many variables, though, so we need to use the extra information that the cost is fixed at 100. This gives us:

$$100 = (\text{cost of sides}) + (\text{cost of bottom}) + (\text{cost of top})$$

Now, the cost of the sides is 2 times the area of the sides, which is:

$$\text{cost of sides} = 2(wh + wh + wh + wh) = 2 \cdot 4wh = 8wh$$

because each side has one side of lengh h and one of length w.

The cost of the base is 2 times the area of the base:

$$\text{cost of base} = 2(w^2)$$

and the cost of the top is 7 times the area of the top:

$$\text{cost of top} = 7(w^2)$$

In total we have:

$$100 = 8wh + 2w^2 + 7w^2 = 8wh + 9w^2$$

Why are we looking at this? We wanted to maximize volume. We wrote an equation for volume, and it had two variables, w and h. We wanted to eliminate one of these variables. We produced the above equation. Now all we need to do is solve in the above equation for either w or h and plug back into our equation for V. We can easily solve for h so we do that:

$$100 = 8wh + 9w^2$$

gives us:

$$100 - 9w^2 = 8wh$$

and dividing by $8w$, we have

$$h = \frac{100}{8w} - \frac{9w^2}{8w} = \frac{25}{2w} - \frac{9}{8}w$$

plugging back into the equation for V gives:

$$V(w) = w^2(\frac{25}{2w} - \frac{9}{8}w) = 25w - \frac{9}{8}w^3$$

Now, we need to know what interval this function is defined on. This is perhaps a bit hard to visualize (and don't stress it too much), but here's the deal: w can get as small as we want, but if we actually let it be 0, we are in trouble, since we must have a cost of 100 at any time - but if w is 0, there is no surface area and therefore no cost! Therefore we cannot have $w = 0$. We therefore have $0 < w$. On the other hand, if w gets big, the entire cost will be in the base and the top. This would give a maximum of w if the cost entirely comes from the top and bottom of the box. We therefore would have:

$$100 = 7w^2 + 2w^2 = 9w^2$$

so $w = \sqrt{100/9} = \sqrt{100}/\sqrt{9} = 10/3$. Therefore we have $w \leq 10/3$ (this really is OK — we actually do have surface area in this case). Therefore the interval is:

$$(0, 10/3]$$

Now for critical points. We take the derivative of

$$V(w) = 25w - \frac{9}{8}w^3$$

and get

$$V'(w) = 25 - \frac{9}{8}3w^2 = 25 - \frac{27}{8}w^2$$

this is always defined. We therefore just need to check when it is 0. Solving $V'(w) = 0$ gives:

$$\frac{27}{8}w^2 = 25$$

and so

$$w^2 = \frac{8 \cdot 25}{27}$$

so $w = \sqrt{\frac{200}{27}}$. Is this in our interval? Let's check. If

$$\sqrt{\frac{200}{27}} < 10/3$$

then we could square both sides and get

$$\frac{200}{27} < \frac{10^2}{3^2} = \frac{100}{9} = \frac{100}{9}\frac{3}{3} = \frac{300}{27}$$

Which checks out. So we do have that this is a critical point.

Now we have to make our sign chart for the derivative. using:

$$V'(w) = 25 - \frac{27}{8}w^2$$

We look to the left of $\sqrt{\frac{200}{27}}$ and check

$$V'(.000000000001) = 25 - \frac{27}{8}(.000000000001)^2 = \text{something very close to } 25$$

is positive. Looking to the right, we check:

$$V'(10/3) = 25 - \frac{27}{8}(10/3)^2 = 25 - \frac{27}{8}\frac{10^2}{3^2} = 25 - \frac{27 \cdot 100}{8 \cdot 9} = 25 - \frac{3 \cdot 25}{2} = 25 - 75/2$$

is negative.

Therefore the function is increasing to the left or our critical point and descreasing on the right. This says that our critical point is actually the maximum, by the first derivative test. □