NAME: Solutions

MATH 2200, Fall 2009
Practice Exam 3

Please hand only only clearly written work, not scratch paper. Clearly mark your final answers for each problem. Partial credit will only be given on problems for which your work is clearly shown.

The only allowable materials for this exam are paper, pens and pencils. No notes, textbooks or calculators will be allowed.

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(1) Suppose that x and y satisfy the equation:

$$x^y = \ln(x)$$

Find the slope of the tangent line to the graph of this equation at the point $(e, 0)$.
[Hint : the change of base formula for exponential functions is: $a^s = e^{s \ln(a)}$]

$$x^y = e^{y \ln x}$$

$$e^{y \ln x} = \ln x$$

$$\frac{d}{dx} e^{y \ln x} = \frac{d}{dx} \ln x$$

$$e^{y \ln x} \cdot \frac{d}{dx}(y \ln x) = \frac{1}{x}$$

$$e^{y \ln x} \cdot \left[\frac{dy}{dx} \ln x + y \frac{d}{dx} \ln x\right] = \frac{1}{x}$$

$$e^{y \ln x} \cdot \left(\frac{dy}{dx} \ln x + y \cdot \frac{1}{x}\right) = \frac{1}{x}$$

$$\frac{dy}{dx} \ln x + \frac{y}{x} = \frac{1}{e^{y \ln x} \cdot x}$$

$$\frac{dy}{dx} \ln x = \frac{1}{e^{y \ln x} \cdot x} - \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{\left(\frac{1}{e^{y \ln x} \cdot x} - \frac{y}{x}\right)}{\ln x}$$

plug in $x = e$, $y = 0$

$$\left.\frac{dy}{dx}\right|_{e,0} = \frac{\left(\frac{1}{e^0 \cdot e} - \frac{0}{e}\right)}{\ln e}$$

$$= \frac{\frac{1}{e} - 0}{1} = \frac{1}{e}$$

So

$$\boxed{\left.\frac{dy}{dx}\right|_{e,0} = \frac{1}{e} = \text{slope of tangent line}}$$

(2) Suppose that x and y are functions of t which satisfy the equation:

$$2x\sin(y) + \cos(x^2) = 0$$

Find an equation for $\frac{dy}{dt}$ in terms of x and y assuming that $\frac{dx}{dt} = 1$

$$\frac{d}{dt}\left(2x\sin y + \cos x^2\right) = \frac{d}{dt}(0)$$

product rule

$$\overbrace{2\frac{dx}{dt}\sin y + 2x\cos y\frac{dy}{dt}} + \overbrace{(-\sin x^2)\cdot 2x\frac{dx}{dt}} = 0$$

$$\frac{dx}{dt} = 1. \text{ so:}$$

$$2\sin y + 2x\cos y\frac{dy}{dt} - \sin x^2 \cdot 2x = 0$$

$$2x\cos y\frac{dy}{dt} = \sin x^2 \cdot 2x - 2x\sin y$$

$$\boxed{\frac{dy}{dt} = \frac{2x\sin x^2 - 2\sin y}{2x\cos y}}$$

(3) A balloon is expanding so that its volume is increasing at a constant rate of 3 $cm^3/sec.$ Using the fact that the volume is given by

$$V = \frac{4}{3}\pi r^3,$$

(a) Find $\frac{dV}{dr}$ when $r = 4$.

$$\frac{d}{dr} V = \frac{d}{dr} \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = \frac{4}{3}\pi 3r^2$$

$$\boxed{\frac{dV}{dr} = 4\pi r^2} \rightarrow r=4 \rightarrow \frac{dV}{dr} = 4\pi (4)^2$$

$$= 4\pi 16 = 64\pi$$

(b) Find $\frac{dr}{dt}$ when $r = 4$.

So $\boxed{\frac{dV}{dr} = 64\pi}$

$$\frac{d}{dt} V = \frac{d}{dt} \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

So $\frac{dr}{dt} = \frac{\left(\frac{dV}{dt}\right)}{4\pi r^2}$ know $r=4$; $\frac{dV}{dt} = 3$

So $\frac{dr}{dt} = \frac{3}{4\pi (4)^2} = \boxed{\frac{3}{64\pi}}$

(4) Suppose that the price (P) and the demand (x) for a commodity are related by the equation

$$2P^2x + \ln(P + x) + \frac{1}{P} = 200$$

(a) Find $\frac{dP}{dx}$ as a function of x and P.

$$\frac{d}{dx}\left(2P^2x + \ln(P+x) + \frac{1}{P}\right) = \frac{d}{dx}(200)$$

$$\left[\frac{d}{dx}(2P^2)\cdot x + 2P^2\cdot\frac{d}{dx}(x)\right] + \left[\frac{1}{P+x}\cdot\frac{d}{dx}(P+x)\right] + \frac{d}{dx}\left(\frac{1}{P}\right) = 0$$

$$4P\cdot\frac{dP}{dx}\cdot x + 2P^2\cdot(1) + \frac{1}{P+x}\left(\frac{dP}{dx} + 1\right) + \left(-\frac{1}{P^2}\right)\frac{dP}{dx} = 0$$

(b) Find $\frac{dx}{dP}$ as a function of x and P.

$$\frac{dx}{dP} = \frac{1}{\left(\frac{dP}{dx}\right)}$$ (next page)

$$4P\frac{dP}{dx}\cdot x + 2P^2 + \frac{1}{P+x}\frac{dP}{dx} + \frac{1}{P+x} - \frac{1}{P^2}\frac{dP}{dx} = 0$$

$$4P\frac{dP}{dx}\cdot x + \frac{1}{P+x}\frac{dP}{dx} - \frac{1}{P^2}\frac{dP}{dx} = -2P^2 - \frac{1}{P+x}$$

$$\frac{dP}{dx}\left[4Px + \frac{1}{P+x} - \frac{1}{P^2}\right] = -2P^2 - \frac{1}{P+x}$$

$$\boxed{\frac{dP}{dx} = \frac{\left(-2P^2 - \frac{1}{P+x}\right)}{\left(4Px + \frac{1}{P+x} - \frac{1}{P^2}\right)}}$$

(4) Suppose that the price (P) and the demand (x) for a commodity are related by the equation

$$2P^2x + \ln(P + x) + \frac{1}{P} = 200$$

(a) Find $\frac{dP}{dx}$ as a function of x and P.

(previous page)

(b) Find $\frac{dx}{dP}$ as a function of x and P.

$$\frac{dx}{dP} = \frac{1}{\left(\frac{dP}{dx}\right)} = \frac{\left(4Px + \frac{1}{P+x} - \frac{1}{P^2}\right)}{\left(-2P^2 - \frac{1}{P+x}\right)}$$

(5) Suppose $f(x) = e^{-x} + \sin x$. Find $f'''(x)$.

$$f'(x) = e^{-x}\frac{d}{dx}(-x) + \cos x$$

$$= e^{-x}(-1) + \cos x = -e^{-x} + \cos x$$

$$f''(x) = -e^{-x} \cdot \frac{d}{dx}(-x) + (-\sin x) = e^{-x} - \sin x$$

$$f'''(x) = e^{-x}\frac{d}{dx}(-x) - \cos x$$

$$= \boxed{-e^{-x} - \cos x}$$

(6) Suppose $f(x) = \ln x$. Find $f'''(x)$.

$$f'(x) = \frac{1}{x} = x^{-1}$$

$$f''(x) = (-1)x^{-1-1} = -x^{-2}$$

$$f'''(x) = -(-2)x^{-2-1} = 2x^{-3} = \boxed{2\frac{1}{x^3}}$$

(7) Calculate the following indefinite integral:

$$\int (e^x + 2x^2 - \sin x)\, dx$$

$$e^x + 2\frac{1}{3}x^3 - (-\cos x) + C$$

$$= \boxed{e^x + \frac{2}{3}x^3 + \cos x + C}$$

(8) Calculate the following indefinite integral:

$$\int x^2 e^{2x^3}\, dx$$

$$u = 2x^3 \qquad du = 6x^2\, dx$$

$$\frac{du}{6} = x^2\, dx$$

$$\int x^2 e^{2x^3}\, dx = \int e^u \frac{du}{6} = \frac{1}{6}\int e^u\, du = \frac{1}{6}e^u + C$$

$$= \boxed{\frac{1}{6}e^{2x^3} + C}$$

(9) Calculate the following indefinite integral:

$$\int \frac{\ln(x)}{x} dx = \int \ln x \cdot \frac{1}{x}\, dx$$

$u = \ln x$

$du = \frac{1}{x} dx$

$$= \int u\, du = \frac{1}{2}u^2 + C$$

$$= \boxed{\frac{1}{2}(\ln x)^2 + C}$$

(10) Calculate the following indefinite integral:

$$\int \left(e^{e^{2x}} e^{2x} + x\right) dx$$

$$\int \left(e^{e^{2x}} e^{2x} + x\right) dx = \int e^{e^{2x}} e^{2x}\, dx + \int x\, dx$$

$$= \underbrace{\int e^{e^{2x}} e^{2x}\, dx}_{} + \frac{1}{2}x^2 + C$$

$u = e^{2x}$

$du = e^{2x} \cdot 2\, dx$

$\frac{du}{2} = e^{2x} dx$

$$= \int e^u \frac{du}{2} = \frac{1}{2}\int e^u du + \frac{1}{2}x^2 + C$$

$$= \frac{1}{2}e^u + \frac{1}{2}x^2 + C$$

$$= \boxed{\frac{1}{2}e^{e^{2x}} + \frac{1}{2}x^2 + C}$$

(11) Calculate the following indefinite integral:

$$\int \frac{x+5}{x} dx$$

$$\int \frac{x+5}{x} dx = \int \frac{x}{x} + \frac{5}{x} dx = \int 1 + 5 \cdot \frac{1}{x} dx$$

$$= x + 5 \ln|x| + C$$

(12) Calculate the following indefinite integral:

$$\int \frac{x}{x+5} dx = \int \frac{u-5}{u} du$$

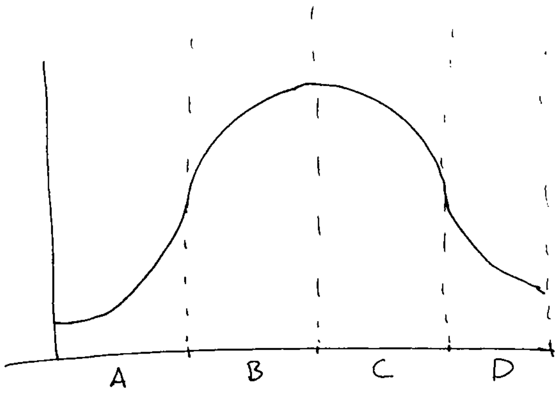
$u = x+5 \longrightarrow u - 5 = x$

$du = (1) dx$

$$\int \frac{u}{u} - \frac{5}{u} du = \int 1 - 5 \cdot \frac{1}{u} du = u - 5 \ln|u| + C$$

$$= (x+5) - 5 \ln|x+5| + C$$

(13) Consider the following graph of a function $f(x)$:



In the spaces below, write either a '+' or a '−' to describe the intervals on which the first and second derivatives of $f(x)$ are positive or negative.

	A	B	C	D
f'(x)	+	+	−	−
f''(x)	+	−	−	+