
MATH 2200, Fall 2009
Practice Final

Solutions

(1) Compute the following derivatives:

(a)

$$\frac{d}{dx} (x^4(3 - 2x^3 - x)x^3)$$

$$= \frac{d}{dx} (3x^7 - 2x^{10} - x^8) = \boxed{21x^6 - 20x^9 - 8x^7}$$

(b)

$$\frac{d}{dy} \left(\frac{e^y - \sin(y)}{y^2 - \ln y} \right)$$

$$\frac{(y - \ln y) \frac{d}{dy} (e^y - \sin y) - (e^y - \sin(y)) \frac{d}{dy} (y^2 - \ln y)}{(y - \ln y)^2}$$

$$(y - \ln y)^2$$

$$\frac{(y - \ln y)(e^y - \cos y) - (e^y - \sin y)(2y - \frac{1}{y})}{(y - \ln y)^2}$$

(c)

$$\frac{d}{dt} (\sin(\sin(e^{\cos t})))$$

$$\boxed{\cos(\sin(e^{\cos t})) \cdot \cos(e^{\cos t}) \cdot e^{\cos t} \cdot (-\sin t)}$$

(d)

$$\frac{d}{ds} \left(\frac{\ln s - s}{s \ln s - s} \right)$$

$$\frac{(s \ln s - s) \frac{d}{ds} (\ln s - s) - (\ln s - s) \frac{d}{ds} (s \ln s - s)}{(s \ln s - s)^2}$$

$$(s \ln s - s)^2$$

$$= \frac{(s \ln s - s) \left(\frac{1}{s} - 1 \right) - (\ln s - s) \left(s \cdot \frac{1}{s} + \ln s - 1 \right)}{(s \ln s - s)^2}$$

can be simplified more.

(2) Compute the following:

(a) Suppose $f(x) = \sin(x) \cos(x)$. What is $f''(x)$?

$$f'(x) = \sin x (-\sin x) + \cos x \cdot \cos x \\ = -\sin^2 x + \cos^2 x$$

$$f''(x) = -2\sin x \cos x + 2\cos x \cdot (-\sin x) \\ = \boxed{-4\sin x \cos x}$$

(b) Suppose $y = f(x)$ is a function such that

$$3y^2x + 2xy - \sin(x+y) = 13$$

Find $\frac{dy}{dx}$ in terms of x and y .

$$\frac{d}{dx}(3y^2x + 2xy - \sin(x+y)) = \frac{d}{dx}(13)$$

(solving: $(6y \frac{dy}{dx} \cdot x + 3y^2 \cdot 1) + (2x \frac{dy}{dx} + xy) - \cos(x+y) \cdot (1 + \frac{dy}{dx}) = 0$)

$$\boxed{\frac{dy}{dx} = \frac{\cos(x+y) - xy - 3y^2}{6xy + 2x - \cos(x+y)}}$$

(3) Consider the function

$$f(x) = e^x$$

Use linear approximation to estimate the value of $f(1.1)$.

$$f(1) = e^1 = e \quad f'(x) = e^x \quad f'(1) = e^1 = e$$

$$f(1.1) \approx f(1) + f'(1)(1.1 - 1)$$

$$= e + e(1.1 - 1)$$

$$= \boxed{e + 0.1e \text{ or } \frac{11}{10}e}$$

(4) Suppose $y = f(x)$ is a function which satisfies the equation

$$x^2 - 4y^3 = 5$$

and such that $f(3) = 1$.

(a) Find the slope of the tangent line to the graph of $f(x)$ at the point $(3, 1)$.

$$\frac{d}{dx}(x^2 - 4y^3) = \frac{d}{dx}(5)$$

$$2x - 12y^2 \frac{dy}{dx} = 0 \quad \rightarrow \quad \frac{dy}{dx} = f'(x) = \frac{-2x}{-12y^2} = \frac{x}{6y^2}$$

$$\frac{dy}{dx} \text{ at } (3, 1) = \frac{3}{6 \cdot 1^2} = \frac{1}{2}$$

(b) Use linear approximation to estimate the value of $f(3 + \frac{2}{17})$.

$$f(3 + \frac{2}{17}) \approx f(3) + f'(3)(3 + \frac{2}{17} - 3) \quad f'(3) = \frac{1}{2}$$

$$= 1 + \frac{1}{2}(\frac{2}{17}) = 1 + \frac{1}{17} = \frac{18}{17}$$

(5) Calculate the following indefinite integrals:

(a)

$$\int (2x^4 - 2x + \frac{1}{x^2}) dx$$

$$= 2 \cdot \frac{1}{5} x^5 - x^2 + \frac{1}{-2+1} x^{-2+1} + C$$

$$= \frac{2}{5} x^5 - x^2 - x^{-1} + C$$

(b)

$$\int x \sin(x^2 - 3) dx$$

$$u = x^2 - 3$$

$$du = 2x dx \quad \frac{1}{2} du = x dx$$

$$= \frac{1}{2} \int \sin u du = \frac{1}{2} (-\cos u) + C = -\frac{1}{2} \cos(u^2 - 3) + C$$

(c)

$$\int \frac{e^{1/x}}{x^2} dx$$

$$u = \frac{1}{x} = x^{-1}$$

$$du = -1x^{-2} dx = -\frac{1}{x^2} dx \Rightarrow -du = \frac{1}{x^2} dx$$

$$= \int e^u (-du) = -\int e^u du = -e^u + C$$

$$= -e^{\frac{1}{x}} + C$$

(d)

$$\int \frac{1}{5} e^{4t-3} dt = \frac{1}{5} \int e^{4t-3} dt$$

$u = 4t - 3$
 $du = 4 dt$

$$= \frac{1}{5} \cdot \int e^u \cdot \frac{1}{4} du$$

$$\frac{1}{4} du = dt$$

$$= \frac{1}{20} \int e^u du = \frac{1}{20} e^u + C = \boxed{\frac{1}{20} e^{4t-3} + C}$$

(e)

$$\int (\cos(x) - 3 \sin(2x)) dx$$

$$= \int \cos x dx - 3 \int \sin 2x dx$$

↙

$$\int \cos x dx = \sin x + C$$

$$\int \sin 2x dx$$

↘

$$u = 2x \quad du = 2 dx$$
$$\frac{1}{2} du = dx$$

$$= \boxed{\sin x - \frac{1}{2} \cdot 3 \cos 2x + C}$$

$$\int \sin u \cdot \frac{1}{2} du = \frac{1}{2} \int \sin u du$$

$$= \frac{1}{2} (-\cos u) + C$$

$$= -\frac{1}{2} \cos 2x + C$$

(6) Suppose y is a function of x such that

$$\frac{dy}{dx} = 4x^2 - 2$$

and $y(1) = 2$. Find y .

$$\int \frac{dy}{dx} dx = \int (4x^2 - 2) dx$$

$$\int dy = \int (4x^2 - 2) dx$$

$$y = 4 \cdot \frac{1}{3} x^3 - 2x + C$$

$$y(1) = 2 \text{ so } 2 = 4 \cdot \frac{1}{3} (1)^3 - 2(1) + C$$

$$2 = \frac{4}{3} - 2 + C = -\frac{2}{3} + C$$

$$f(x) = 4 \cdot \frac{1}{3} x^3 - 2x + \frac{8}{3}$$

$$C = 2 + \frac{2}{3} = \frac{8}{3}$$

(7) Suppose y is a function of x such that

$$y \frac{dy}{dx} = 4x^2 - 2$$

and $y(1) = 2$. Find y .

$$\int y \frac{dy}{dx} dx = \int (4x^2 - 2) dx$$

$$\frac{1}{2} y^2 = \frac{4}{3} x^3 - 2x + C$$

$$y^2 = \frac{8}{3} x^3 - 4x + C$$

$$y = \sqrt{\frac{8}{3} x^3 - 4x + C}$$

$$2 = y(1) = \sqrt{\frac{8}{3} (1) - 4(1) + C} = \sqrt{-\frac{4}{3} + C}$$

$$2^2 = 4 = -\frac{4}{3} + C$$

$$C = 4 + \frac{4}{3} = \frac{16}{3}$$

$$y = \sqrt{\frac{8}{3} x^3 - 4x + \frac{16}{3}}$$

(8) Suppose y is a function of x such that

$$\frac{(y+3)}{y} \frac{dy}{dx} = \frac{y \sin(2x)}{y}$$

and $y(0) = 3$. Find y .

$$\int \frac{(y+3)}{y} \frac{dy}{dx} dx = \int \sin 2x dx$$

$$\int (1 + 3\frac{1}{y}) dy = \int \sin 2x dx \quad \swarrow (\text{via } u=2x)$$

$$y + 3 \ln y = -\frac{1}{2} \cos 2x + C$$

← (too hard to solve for y .
Sorry)

(9) Suppose y is a function of x such that

$$\frac{dy}{dx} = 4(y-3)^3 - \sin(x+y-7) + 3$$

and $y(4) = 3$.

(a) Find the slope of the tangent line to the graph of the function y at the point $(4, 3)$.

$$\begin{aligned} \frac{dy}{dx} \text{ at } (4, 3) & \text{ is} \\ & = 4(3-3)^3 - \sin(4+3-7) + 3 \\ & = 4(0) - \sin(0) + 3 = \boxed{3} \end{aligned}$$

(b) Use linear approximation to estimate the value of $y(4 + \frac{1}{10})$.

$$\begin{aligned} y(4 + \frac{1}{10}) & \approx y(4) + y'(4)(4 + \frac{1}{10} - 4) \\ & = 3 + 3(\frac{1}{10}) = 3 + \frac{3}{10} = \boxed{\frac{33}{10}} \end{aligned}$$

(10) Consider the function

$$f(x) = \ln x + \frac{7}{x} - 3 \quad \text{on the interval } [1, \infty)$$

(a) Find all critical points of $f(x)$ on this interval (or write 'none').

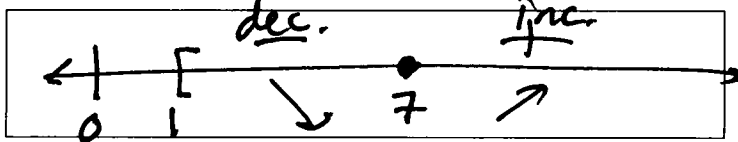
Critical points:

$$\boxed{7}$$

$$f'(x) = \frac{1}{x} - \frac{7}{x^2} = \frac{x-7}{x^2}$$

$$0 \text{ at } x=7$$

(b) Construct sign chart for the derivative of $f(x)$ in the space below, illustrating where the function $f(x)$ is increasing and decreasing.



Show your work below:

$$f'(2) = \frac{2-7}{2^2} < 0$$

$$f'(10) = \frac{10-7}{10^2} > 0$$

(c) Find the minimum value obtained by $f(x)$ on the given interval:

minimum value :

$$\boxed{\ln 7 - 4}$$

$$\begin{aligned} \text{min at } x=7 \text{ is } f(7) &= \ln 7 - \frac{7}{7} - 3 \\ &= \ln 7 - 4 \end{aligned}$$

- (11) A farmer would like to set up a rectangular enclosure of fencing at a fixed cost of \$100. If three of the sides of the enclosure cost \$2 per linear foot of fencing, and the fourth side costs \$5 per linear foot, find the dimensions of the enclosure which maximize the total area at this cost.

(a) What is the quantity which you are trying to maximize?

$$\text{area} = A$$

Write an equation for the quantity to be maximized, as a function of only one variable, and state what the meaning of this variable is (i.e. radius, diameter, height, etc.).

Equation : $A = l \left(\frac{100 - 7l}{4} \right)$

Use the space below to show your work.

$$A = lw \quad l = \text{length} \quad w = \text{width}$$

$$\begin{aligned} \text{Cost} = 100 &= 2w + 2w + 2l + 5l \\ &= 4w + 7l \quad \text{so} \end{aligned}$$

$$w = \frac{100 - 7l}{4}$$

(b) What is the interval on which this function is defined?

$$\left[0, \frac{100}{7} \right]$$

You may justify your answer below (only for partial credit if above answer is incorrect).

max length



$$l : \text{cost} = 7l = 100$$

$$\text{so } l = \frac{100}{7}$$

(c) Find all critical points of this function on the interval you have found.

$$\boxed{50/7}$$

Use the space below to show your work.

$$A(l) = l \left(\frac{100 - 7l}{4} \right) = 25l - \frac{7}{4}l^2$$

$$A'(l) = 25 - \frac{7}{2}l = 0$$

↓

$$l = \frac{50}{7}$$

(d) Find the dimensions of the fence (length, width) which maximize the total area.

Dimensions:

$$\boxed{\text{length} = 50/7 \quad \text{width} = 25/2}$$

Use the space below to show your work.

$$l = 50/7 \quad \text{so} \quad w = \frac{100 - 7l}{4}$$

$$= \frac{100 - 7(50/7)}{4}$$

$$= \frac{100 - 50}{4} = \frac{50}{4} = \frac{25}{2}$$

(12) Suppose the demand of a given commodity is related to its price by the equation

$$D \ln(D/10) + 1 = \frac{1}{p^2}$$

where D is the demand and p is the price, and recall that the revenue may be expressed as

$$R = Dp.$$

(a) Find the rate of change of the revenue with respect to price when $p = 10$ and $D = 10$

$$\frac{dR}{dp} = \frac{dD}{dp} \cdot p + D$$

find $\frac{dD}{dp}$: $\frac{d}{dp}(D \ln(D/10) + 1) = \frac{d}{dp}\left(\frac{1}{p^2}\right) = \frac{d}{dp}(p^{-2})$

$$\frac{dD}{dp} \ln(D/10) + D \cdot \frac{1}{(D/10)} \cdot \frac{1}{10} \frac{dD}{dp} = -2p^{-3}$$

$$\frac{dD}{dp} (\ln D/10) + \frac{dD}{dp} = -2 \frac{1}{p^3}$$

(b) Use linear approximation to estimate the revenue when the price is 11.

$$R|_{p=11} \approx R|_{p=10} + \left. \frac{dR}{dp} \right|_{p=10} (11-10)$$

$$= 100 + \left(10 - \frac{1}{50}\right)(1)$$

$$= \boxed{110 - \frac{1}{50}} = \boxed{109.98}$$

$$p=10, D=10$$

$$\boxed{R = 10 \cdot 10 = 100}$$

$$D=10 \quad p=10$$

$$\left. \frac{dD}{dp} \right|_{p=10} (\ln 1) + \frac{dD}{dp} = 2 \frac{-1}{10^3}$$

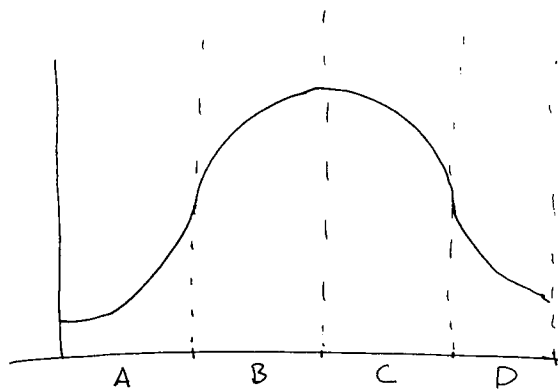
$$\frac{dD}{dp} = -\frac{2}{1000}$$

$$\frac{dR}{dp} = \frac{dD}{dp} \cdot p + D$$

$$= -\frac{2}{1000} \cdot 10 + 10$$

$$= -\frac{2}{100} + 10 = \boxed{10 - \frac{1}{50}}$$

(13) Consider the following graph of a function $f(x)$:



In the spaces below, write either a '+' or a '-' to describe the intervals on which the first and second derivatives of $f(x)$ are positive or negative.

	A	B	C	D
$f'(x)$	+	+	-	-
$f''(x)$	+	-	-	+