MATH 2200, Fall 2009
Exam 1

(1) Find the derivatives of the following functions:

(a)

$$f(x) = (2x^4 - x^3 + 7x)^7$$

SOLUTION

$$f'(x) = 7(2x^4 - x^3 + 7x)^6 \frac{d}{dx}(2x^4 - x^3 + 7x)$$
$$= 7(2x^4 - x^3 + 7x)^6(8x^3 - 3x^2 + 7)$$

(b)

$$f(x) = \frac{(3x^3 - 8x^2)(1 + x + x^2)}{4x - 1}$$

SOLUTION

$$f'(x) = \frac{(4x-1)\frac{d}{dx}\big((3x^3 - 8x^2)(1 + x + x^2)\big) - (3x^3 - 8x^2)(1 + x + x^2)\frac{d}{dx}\big(4x - 1\big)}{(4x-1)^2}$$
$$= \frac{(4x-1)\big((3x^3 - 8x^2)\frac{d}{dx}(1 + x + x^2) + (1 + x + x^2)\frac{d}{dx}(3x^3 - 8x^2)\big) - (3x^3 - 8x^2)(1 + x + x^2)(4)}{(4x-1)^2}$$
$$= \frac{(4x-1)\big((3x^3 - 8x^2)(1 + 2x) + (1 + x + x^2)(9x^2 - 16x)\big) - (3x^3 - 8x^2)(1 + x + x^2)(4)}{(4x-1)^2}$$

(c)

$$f(x) = \frac{3x - 3}{\sqrt{x^2 + 1}}$$

SOLUTION

$$f'(x) = \frac{\left(\sqrt{x^2+1}\right)\frac{d}{dx}(3x - 3) - (3x - 3)\frac{d}{dx}\left(\sqrt{x^2+1}\right)}{x^2 + 1}$$
$$= \frac{\left(\sqrt{x^2+1}\right)(3) - (3x - 3)\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\frac{d}{dx}(x^2 + 1)}{x^2 + 1}$$
$$= \frac{\left(\sqrt{x^2+1}\right)(3) - (3x - 3)\frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)}{x^2 + 1}$$

(d)

$$f(x) = \sqrt{x^2 + \sqrt{1 - x}}$$

SOLUTION

$$f'(x) = \frac{1}{2}\left(x^2+\sqrt{1-x}\right)^{-\frac{1}{2}}\frac{d}{dx}\left(x^2+\sqrt{1-x}\right)$$
$$= \frac{1}{2}\left(x^2+\sqrt{1-x}\right)^{-\frac{1}{2}}\left(2x+\frac{1}{2}(1-x)^{-\frac{1}{2}}\frac{d}{dx}(1-x)\right)$$
$$= \frac{1}{2}\left(x^2+\sqrt{1-x}\right)^{-\frac{1}{2}}\left(2x+\frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)\right)$$

(2) Use the definition of the derivative to find the derivatives of the following functions:

(a)

$$f(x) = 2x + 4$$

SOLUTION

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{\big(2(x+h)+4\big)-(2x+4)}{h}$$
$$= \lim_{h\to 0}\frac{2x+2h+4-2x-4}{h}$$
$$= \lim_{h\to 0}\frac{2h}{h} = \lim_{h\to 0} 2 = 2$$

So, $f'(x) = 2$.

(b)

$$f(x) = 5x^2 + 1$$

SOLUTION

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{(5(x+h)^2+1)-(5x^2+1)}{h}$$
$$= \lim_{h\to 0}\frac{5(x^2+2xh+h^2)+1-5x^2-1}{h}$$
$$= \lim_{h\to 0}\frac{5x^2+10xh+5h^2+1-5x^2-1}{h}$$
$$= \lim_{h\to 0}\frac{10xh+5h^2}{h} = \lim_{h\to 0} 10x+5h = 10x$$

So $f'(x) = 10x$.

(3) Find an equation for the tangent line to the graph of the following functions at the given x values:

(a)

$$f(x) = 5x^2 + 2 \quad \text{at } x = 1$$

SOLUTION We first find $f'(x)$ via the power rule:

$$f'(x) = 5 \cdot 2x + 0 = 10x$$

We then find the slope of the tangent line by plugging in the given x-value of 1 into $f'(x)$:

$$\text{slope} = f'(1) = 10(1) = 10$$

Next, we need a point on the tangent line in order to use point-slope form. Since we are given that the x-value of our point on the graph is $x = 1$, our tangent line will pass through the point on the graph of $f(x)$ with coordinates

$$(1, f(1)) = (1, 5(1)^2 + 2) = (1, 7)$$

Finally, we use the point slope form to obtain an equation for a line passing through $(1, 7)$ with a slope of 10:

$$y - 7 = 10(x - 1)$$

(b)

$$f(x) = \frac{1}{x} \quad \text{at } x = 3$$

SOLUTION We first find $f'(x)$ via the power rule, writing $f(x) = x^{-1}$ (Note: this is one of many ways in which you can take this derivative — for example, one might also use the quotient or reciprocal rule):

$$f'(x) = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

We then find the slope of the tangent line by plugging in the given x-value of 3 into $f'(x)$:

$$\text{slope} = f'(3) = -\frac{1}{3^2} = -\frac{1}{9}$$

Next, we need a point on the tangent line in order to use point-slope form. Since we are given that the x-value of our point on the graph is $x = 3$, our tangent line will pass through the point on the graph of $f(x)$ with coordinates

$$(3, f(3)) = (3, \frac{1}{3})$$

Finally, we use the point slope form to obtain an equation for a line passing through $(3, \frac{1}{3})$ with a slope of $\frac{1}{9}$:

$$y - \frac{1}{3} = \frac{1}{9}(x - 3)$$

(4) A ball is thrown straight upwards at a speed of 40 *meters/sec*. Its height as a function of time is given as

$$y(t) = -5t^2 + 40t$$

(a) After how many seconds is the velocity of the ball equal to 0?
SOLUTION The velocity is the derivative of the height function. Therefore it is given by

$$\begin{aligned} y'(t) &= -5 \cdot 2t + 40 \\ &= -10t + 40 \end{aligned}$$

To solve the question, we need to solve for t when $y'(t) = 0$. This gives:

$$\begin{aligned} 0 &= -10t + 40 \\ 10t &= 40 \\ t &= 4 \end{aligned}$$

Therefore the ball has a velocity of 0 after 4 seconds have elapsed.

(b) What is the height of the ball when its velocity is 0?
SOLUTION By the previous part of the problem, the velocity of the ball is 0 exactly when $t = 4$. Therefore we just need to find the height when $t = 4$. This is just $y(4) = -5(4)^2 + 40(4) = -5(16) + 160 = 80$.

(c) What is the average velocity of the ball between the time at which it is first thrown and the time at which its velocity is 0?
SOLUTION The average velocity is given by the total change in height divided by the change in time. Since the ball has a velocity of 0 exactly after 4 seconds, the initial time is 0 and the final time is 4. Therefore the average velocity is:

$$\frac{f(4) - f(0)}{4 - 0} = \frac{80 - 0}{4} = 20$$

(d) After how many seconds does the ball hit the ground?
SOLUTION We simply need to solve for when $y = 0$.

$$\begin{aligned} 0y(t) &= -5t^2 + 40t \\ 0 &= t(-5t + 40) = -5t(t - 8) \end{aligned}$$

and so we find that $y(t) = 0$ exactly when $t = 0$ or $t = 8$. Since $t = 0$ corresponds to the moment the ball was thrown, we see that $t = 8$ gives us the time at which the ball landed back on the ground.

(5) Suppose $f(x)$ is a function such that

$$\frac{d}{dx} f(x) = \frac{1}{x}$$

Find $\frac{d}{dx} f(x^2)$.

SOLUTION We use the chain rule on $f(x^2)$, considering f to be the 'outer' function and x^2 to be the 'inner' function. This gives:

$$\frac{d}{dx} f(x^2) = f'(x^2) \frac{d}{dx}(x^2) = \frac{1}{x^2}(2x) = \frac{2x}{x^2} = \frac{2}{x}$$

(6) Suppose $f(x)$ and $g(x)$ are functions such that $f(g(x)) = x$. Take the derivative of both sides of this equation and use the chain rule to show that

$$g'(x) = \frac{1}{f'(g(x))}$$

Show all your work.

SOLUTION taking derivatives of both sides of the equation $f(g(x)) = x$ yields:

$$f'(g(x))g'(x) = \frac{d}{dx} x = 1$$

Now, dividing both sides by $f'(g(x))$, we find

$$g'(x) = \frac{1}{f'(g(x))}$$

as desired.

(7) Let $f(x) = x^4$.

(a) Find an equation for the tangent line to the graph of $f(x)$ at $x = 3$.
SOLUTION When $x = 3$, the y value is $y = f(3) = 3^4 = 81$. Also, the derivative of $f(x)$ is $f'(x) = 4x^3$. Therefore the slope of the tangent line at $x = 3$ is

$$f'(3) = 4(3)^3 = 4(27) = 108.$$

This gives the equation for the tangent line as

$$y - 81 = 108(x - 3)$$

or

$$y = 108(x - 3) + 81$$

(b) Use the equation for the tangent line above to estimate 3.2^4.
SOLUTION
We simply plug 3.2 into the equation for the tangent line above:

$$3.2^4 = f(3.2) \approx 108(3.2 - 3) + 81 = 108(.2) + 81 = 108(2/10) + 81 = \frac{108}{5} + 81$$