Initial Value Problems

ex: $\frac{dy}{dx} = 6y$ $\quad y(0) = 3$

solve for y.

$$\frac{1}{y}\frac{dy}{dx} = 6$$

$$\int \frac{1}{y}\frac{dy}{\cancel{dx}}\cancel{dx} \quad \int 6\,dx$$

$$\int \frac{1}{y}\,dy = \int 6\,dx$$

$$\ln y = 6x + C$$

solve for y

$$y = e^{6x+C}$$

$$= e^{6x}e^{C} = C'e^{6x}$$

last time

$y(0)=3 \Rightarrow y(0) = 3 = C'e^{6(0)} = C'$

I.V.P.

$C' = 3$

$$\boxed{y = 3e^{6x}}$$

$$\frac{dy}{dx} - 6y = 3 \qquad y(0) = 0$$

$$F(y)\frac{dy}{dx} = G(x) \longleftarrow \text{Want.}$$

$$\int\left(\frac{dy}{dx} - 6y\right)dx = \int 3\,dx$$

$$\int 1\frac{dy}{\cancel{dx}}\,\cancel{dx} - \underset{?}{\int 6y\,dx} = \int 3\,dx$$

$$y \qquad\qquad = 3x$$

Stuck.

$$\frac{dy}{dx} - 6y = 3 \rightarrow \frac{dy}{dx} = 3 + 6y$$

$$\frac{1}{3+6y}\frac{dy}{dx} = 1$$

$\int$ ____ dx both sides

$$\int \frac{1}{3+6y}\frac{dy}{\cancel{dx}}\cancel{dx} = \int 1\,dx$$

$$\int \frac{1}{3+6y}\,dy = \int 1\,dx$$

$$= x$$

$$\int \frac{1}{3+6y}\,dy = \int \frac{1}{u}\cdot\frac{1}{6}\,du = \frac{1}{6}\int\frac{1}{u}\,du$$

$u = 3+6y$

$$= \frac{1}{6}\ln u + C$$

$\frac{du}{dy} = 6 \Rightarrow du = 6\,dy$

$\frac{1}{6}du = dy$

$$= \frac{1}{6}\ln(3+6y) + C$$

$$\frac{1}{6}\ln(3+6y) = x + C$$

$$\ln(3+6y) = 6x + 6C$$

$$3+6y = e^{6x+6C}$$

$$6y = e^{6x+6C} - 3$$

$$y = \frac{1}{6}\left(e^{6x+6C} - \frac{3}{\cancel{6}}\right)$$

Now use $y(0) = 0$

$y(0) = 0$

$0 = y(0) = \frac{1}{6}\left(e^{0+6c} - 3\right)$

$0 = e^{6c} - 3$

$3 = e^{6c}$

$$y = \frac{1}{6}\left(e^{6x+6c} - 3\right)$$

$$= \frac{1}{6}\left(e^{6x}\underbrace{e^{6c}}_{3} - 3\right)$$

$$\boxed{y = \frac{1}{6}\left(e^{6x} \cdot 3 - 3\right)}$$

Practice

1. $\frac{dy}{dx} = 3xy^2 \qquad y(1) = 2$

2. $\frac{dy}{dx} = \frac{1+x}{xy^5} \qquad y(1) = 2$

1. $\frac{1}{y^2}\frac{dy}{dx} = 3x$

$$\int \frac{1}{y^2}\frac{dy}{\cancel{dx}}\cancel{dx} = \int 3x\,dx$$

$$\int \frac{1}{y^2}\,dy = \int 3x\,dx$$

$$\int y^{-2}\,dy = \int 3x\,dx$$

$$\frac{1}{-2+1}y^{-2+1} = 3\,\frac{1}{2}x^2 + C$$

$$-\frac{1}{y} = \frac{3}{2}x^2 + C$$

$$\frac{1}{y} = -\left(\frac{3}{2}x^2 + C\right)$$

$$y = \frac{1}{-\frac{3}{2}x^2 - C}$$

$y(1) = 2$ (let $x = 1$, $y = 2$)

$$2 = \frac{1}{-\frac{3}{2}(1)^2 - C} = \frac{1}{-\frac{3}{2} - C}$$

$$\frac{1}{2} = -\frac{3}{2} - C$$

$$2 = \frac{1}{2} + \frac{3}{2} = -C$$

$$\boxed{y = \frac{1}{-\frac{3}{2}x^2 + 2}}$$

2. $\dfrac{dy}{dx} = \dfrac{1+x}{xy^5}$ $\quad y(1)=2$

$$y^5 \frac{dy}{dx} = \frac{1+x}{x}$$

$$\int y^5 \frac{dy}{\cancel{dx}} \cancel{dx} = \int \frac{1+x}{x}\,dx = \int\left(\frac{1}{x}+1\right)dx$$

$$\frac{1}{6}y^6 = \ln x + x + C$$

$$y^6 = 6(\ln x + x + C)$$

$$y = \sqrt[6]{6(\ln x + x + C)}$$

$y(1)=2$

$y=2 \quad x=1$

$$2 = \sqrt[6]{6(\underbrace{\ln 1}_{0} + 1 + C)}$$

$$64 = 2^6 = 6(1+C)$$

$$\frac{32}{3} = 1+C \qquad C = \frac{32}{3} - 1 = \frac{29}{3}$$

$$\boxed{y = \sqrt[6]{6\left(\ln x + x + \frac{29}{3}\right)}}$$

Modelling stock prices

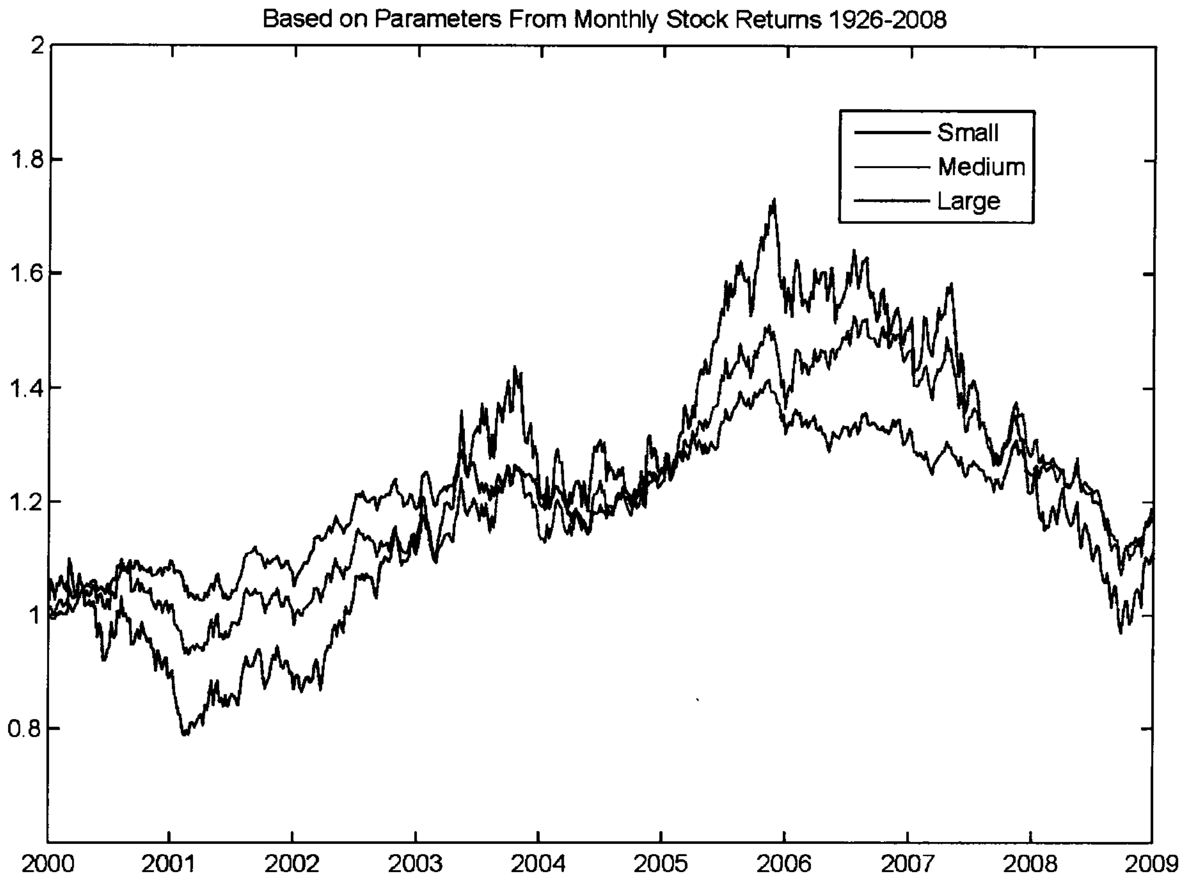
Major difference:

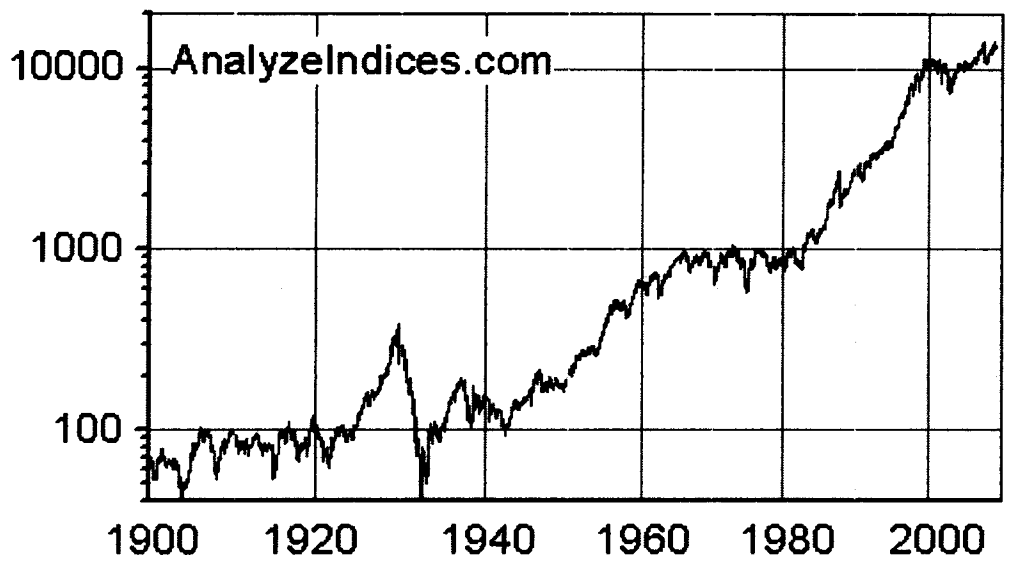
some aspects of our functions will be unknown/random

Basic model for stocks

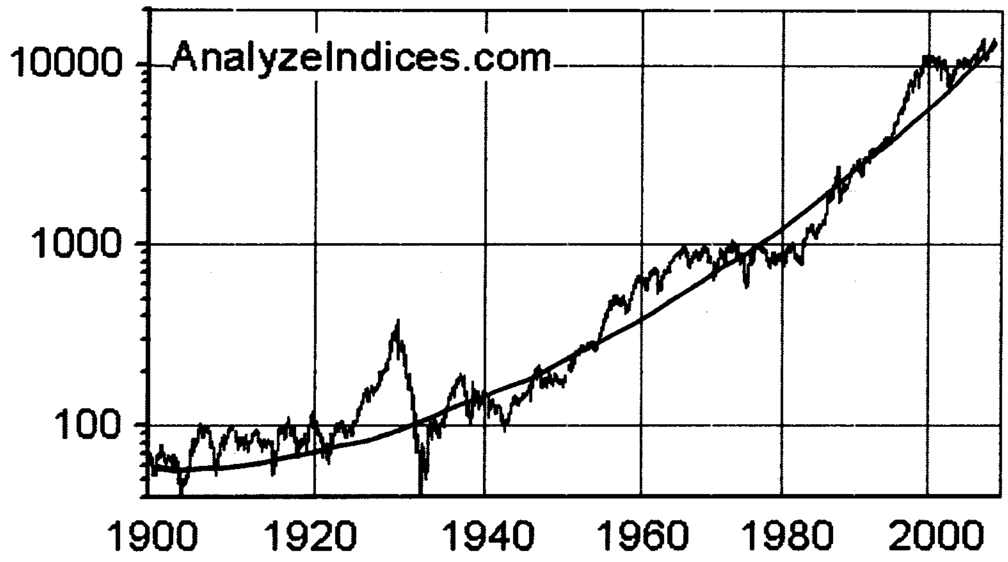
"Brownian motion w/ drift"

Simulations of Small, Medium, and Large-Cap Stock Prices





(Dow Jones History)


AnalyzeIndices.com
10000
1000
100
1900
1920
1940
1960
1980
2000

"
exp. growth.

$$\frac{dS}{dt} = \mu S \leftarrow \text{standard intrest eqn}$$

$$dS = \mu S \, dt$$

divide by dt:

$$\frac{dS}{dt} = \mu S \overset{1}{\frac{dt}{dt}} + \sigma S \cdot \overset{\substack{\text{random stuff} \\ \downarrow}}{\frac{dW}{dt}}$$

$$\frac{dS}{dt} = S\left(\mu + \sigma \underset{\text{``W'"}}{\frac{dW}{dt}}\right)$$

S = price of stock

$$dS = \underset{\substack{\uparrow \\ \text{int. rate}}}{\mu} \underset{\substack{\uparrow \\ \text{exp-growth}}}{S} dt + \underset{\substack{\uparrow \\ \text{volatility}}}{\sigma} S \underset{\substack{\uparrow \\ \text{noise}}}{dW}$$

says stock price varies in proportion to current value in a random way

Can "solve"

roughly, divide by dt, rearrge & $\int$

— need special tools to $\int$ random processes "Ito integral"

like a "path integral" used in modern particle physics.

$$S = S_0 e^{\left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma \underset{\substack{\uparrow \\ \text{random}}}{W_t} \right)}$$

Usefullness: quantify risk & effects of volatility

- why quantify risk?

If can quantify risk better than everyone else, can make lots of cash.

Option

House want to buy for $400,000

pay $2,000 for option to buy at this price within a month.

Scenario 1: House is infested by giant rats.

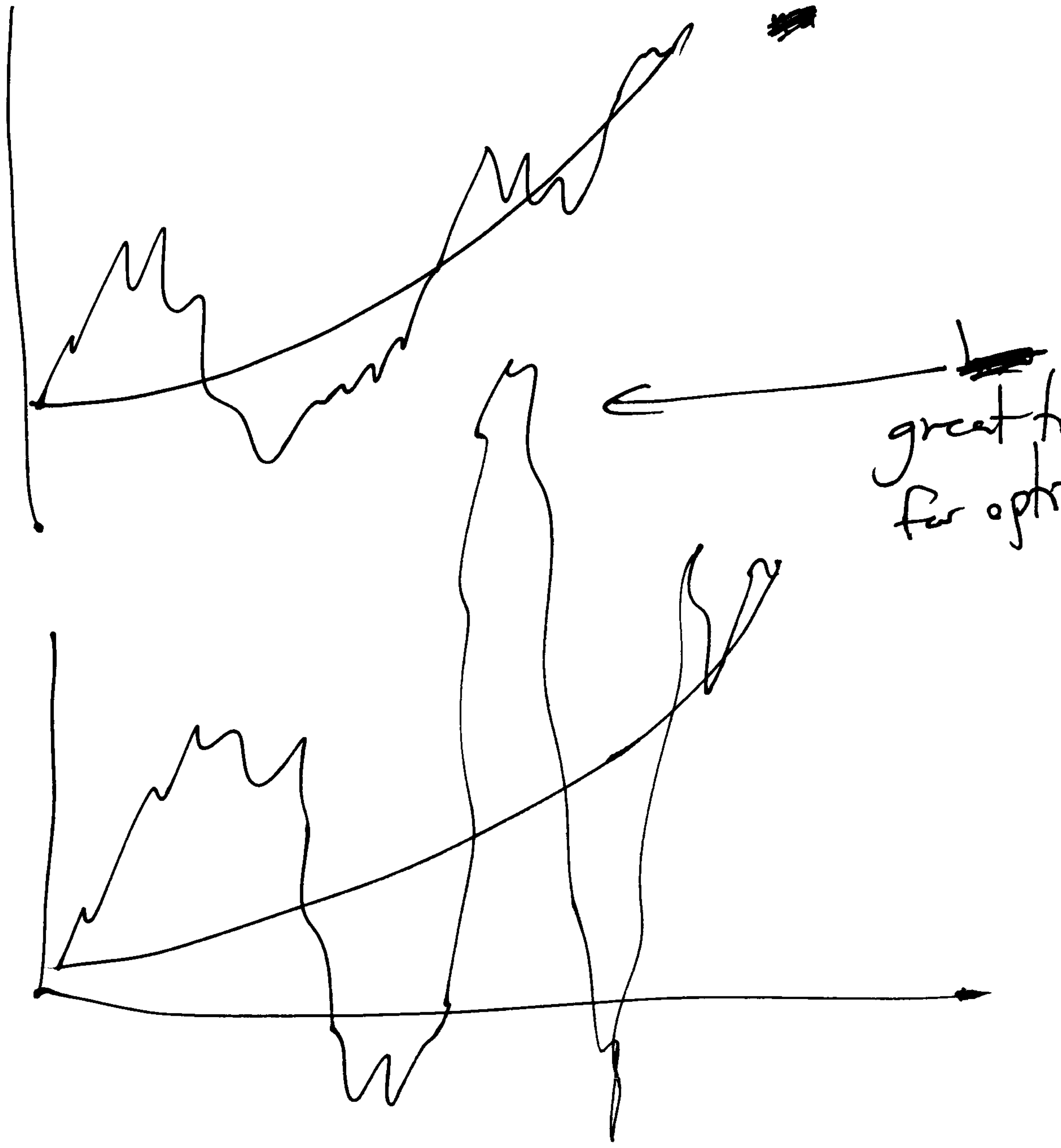
lose $2,000

Scenario 2: Prior home of Elvis's mother

house goes up in value $2,000,000

Big question How to price an option.

more volatility = ~~more~~ higher price



great time for option.