

QUADRATIC FORMS AND THE PERIOD-INDEX PROBLEM

DANIEL KRASHEN

The main purpose of this paper is to notice that we may derive fundamental results concerning the period and the index as easy corollaries of deep recent results in the theory of quadratic forms.

Given a field k , we ask the question : how complicated can the elements of the cohomology groups $H^m(k, \mu_p)$ or $H^m(k, \mathbb{G}_m)$ be? In order to quantify this, we use two invariants of cohomology classes: the index and the period. In the case of the Brauer group ($H^2(k, \mathbb{G}_m)$) these invariants have been studied for various fields k . In particular, a fundamental open question is the relationship between these two invariants and the cohomological dimension of the field. We present an extremely hopeful conjecture concerning this relationship below.

Let k be a field with $\text{char}(k) \neq 2$. The Brauer group $Br(k) = H^2(k, \mathbb{G}_m)$ is a torsion abelian group whose elements correspond to isomorphism classes of k central division algebras. The index of an element $\alpha \in Br(k)$ is defined to be the square root of the k -vector space dimension of the corresponding division algebra, and the period of α is defined to be its order as an element of group $Br(k)$.

It is a classical fact that the period divides the index and that these two numbers have the same prime factors. Therefore one may see that $\text{ind}(\alpha) | \text{per}(\alpha)^r$ for some r which may depend on the class α . We recall the following “folk conjecture,” as stated in [Lie]:

Conjecture 1. *Suppose k is a C_n field. If $\alpha \in Br(k)$ then we have $\text{ind}(\alpha) | \text{per}(\alpha)^{n-1}$.*

This certainly holds for C_1 fields (i.e. $Br(k) = 0$ for such a field). At present, this conjecture has been verified for function fields of surfaces over algebraically closed fields (see [dJ04] and [Lie]). Lieblich has also shown that $\text{ind} | \text{per}^3$ for surfaces over finite fields (the conjectural bound would be $\text{ind} | \text{per}^2$). Saltman has shown that for k the function field of a curve over a p -adic field, $\text{ind} | \text{per}^2$. At present it is not known whether the conjecture holds for an arbitrary C_n field when $n > 1$. In fact, for $n > 1$ it is not known if there exists a finite integer r such that $\text{ind}(\alpha) | \text{per}(\alpha)^r$ for every C_n field k .

In this paper, we explore the question as posed above as well as the same question for elements in higher cohomology groups $H^m(k, \mathbb{G}_m)$. For these groups, we make the following conjecture (see definition 3 for an explanation of the terminology):

Conjecture 2. *Suppose k is a C_n field. If $\alpha \in H^m(k, \mathbb{G}_m)$, $m > 1$, then we have $\text{ind}(\alpha) | \text{per}(\alpha)^{n-m+1}$.*

This conjecture is justified by the proposition ??, where we show that in the case that $k = F((t_1))((t_2)) \cdots ((t_n))$, where F is algebraically closed, then the above bound is sharp.

In this generality we obtain partial evidence for this by showing that if α has 2-power order then for every n there exists an r such that if k is a C_n field, $\text{ind}(\alpha) | \text{per}(\alpha)^r$ (definitions of these terms are given below). This r is given explicitly in theorem 4. In particular, we have:

- If k is a C_n field and $\alpha \in H^n(k, \mathbb{G}_m)$, then $\text{ind}(\alpha) = \text{per}(\alpha)$.
- If k is a C_n field and $\alpha \in H^{n-1}(k, \mathbb{G}_m)$, then $\text{ind}(\alpha) | \text{per}(\alpha)^n$.

Note that the first case coincides with the result of deJong ([dJ04]) for surfaces over an algebraically closed field, and the second coincides with the result of Lieblich for surfaces over a finite field ([Lie]).

We note also that results of Elman and Lam suggest the existence of a partial converse to these statements. Namely, their results show that if $\text{per}(\alpha) = \text{ind}(\alpha)$ for all $\alpha \in \text{Br}(k)$ then the u invariant $u(k) \leq 8$ (see [Pfi95], page 125). This essentially says that k behaves like a C_3 field with respect to quadratic forms.

We prove our results by observe that if k has a finite u invariant (for example, if k is C_n for some n), then via the Milnor Conjecture (proved by Voevodsky in [Voe03a, Voe03b]), one may give a bound on the degree of a field extension necessary to split a given 2-power torsion cohomology class as a function of its period. Knowledge of the possible dimensions of anisotropic forms of small size given by [Kar04], allows one to further improve these bounds.

The following notation will be used throughout:

Definition 3. *Let $\alpha \in H^m(k, \mathbb{G}_m)$. We define the index of α , written $\text{ind}(\alpha)$ to be $\text{GCD}\{[K : k] | \alpha_K = 0\}$, where K/k is a separable extension. The period of $\text{per}(\alpha)$ is defined to be its order in the group $H^m(k, \mathbb{G}_m)$.*

We similarly define the index for elements in $H^m(k, \mu_2)$.

We are particularly in the case that $\alpha \in H^m(k, \mathbb{G}_m)_2$, which we define to be the subgroup of the cohomology group consisting of elements of 2-power order. The standard restriction-corestriction argument shows that $\text{per}(\alpha) | \text{ind}(\alpha) = 2^s$ for some s .

Theorem 4. *Suppose k is a nonreal field of u invariant $u(k) \leq 2^n$ (for example, a C_n field), and $\alpha \in H^m(k, \mathbb{G}_m)_2$. Then*

- (1) *If $m > n$ then $\alpha = 0$.*
- (2) *If $m = n$ then $\text{ind}(\alpha) = \text{per}(\alpha)$.*
- (3) *If $m < n$, then $\text{ind}(\alpha) | \text{per}(\alpha)^{2^{n-1}-2^m+m+1}$.*

If k is formally real with the same hypotheses, then we have:

- (1) *If $m > n + 2$ then $\text{ind}(\alpha) = \text{per}(\alpha)$.*
- (2) *If $m = n + 2$ then $\text{ind}(\alpha) | \text{per}(\alpha)^2$.*
- (3) *If $m < n + 2$, then $\text{ind}(\alpha) | \text{per}(\alpha)^{2^{n-1}-2^m+m+2}$.*

This is an immediate consequence of the following Lemma 5 combined with Proposition 7.

Lemma 5. *Suppose k is a field such that for every finite extension L/k and every $\alpha \in H^n(k, \mu_2)$, $\text{ind}(\alpha) | 2^s$. Then for every 2-power torsion element $\beta \in H^n(k, \mathbb{G}_m)$, we have $\text{ind}(\alpha) | \text{per}(\alpha)^s$.*

Proof. We may assume without loss of generality that k is prime to 2 closed. Suppose α is 2^r -torsion and let $\beta = 2^{r-1}\alpha$. Since β is 2-torsion, it is the image of an element in $H^n(k, \mu_2)$. Therefore, there is a field extension L/k of degree dividing 2^s such that $\beta_L = 0$. But by induction α_L possesses a degree $(2^{r-1})^s$ splitting field which is therefore a degree $(2^r)^s$ splitting field regarded as an extension of k . \square

In the remainder of the paper, we will have use of the following theorem proved by Karpenko, originally conjectured by Vishik:

Theorem 6 ([Kar04], conjecture 1.1). *If ϕ is an anisotropic quadratic form such that $\phi \in I^n \subset W(k)$, where I is the fundamental ideal of the Witt ring, and $\dim(\phi) < 2^{n+1}$, then $\dim(\phi) = 2^{n+1} - 2^i$ for $1 \leq i \leq n + 1$.*

Proposition 7. *Suppose k is a nonreal field of u invariant $u(k) \leq 2^n$ (for example, a C_n field), and $\alpha \in H^m(k, \mu_2)$. Then*

- (1) *If $m > n$ then $\alpha = 0$. In other words, $cd_2(k) \leq n$.*
- (2) *If $m = n$ then $\text{ind}(\alpha) | 2$.*
- (3) *If $m < n$, then $\text{ind}(\alpha) | 2^{2^{n-1}-2^m+m+1}$.*

If k is formally real with the same hypotheses, then we have:

- (1) *If $m > n + 2$ then $\text{ind}(\alpha) | 2$.*
- (2) *If $m = n + 2$ then $\text{ind}(\alpha) | 4$.*
- (3) *If $m < n + 2$, then $\text{ind}(\alpha) | 2^{2^{n-1}-2^m+m+2}$.*

Proof. Note that the statements for k formally real follow immediately from the statements for a nonreal field by extending scalars to $k(\sqrt{-1})$, since by [Pfi95], page 114, $u(k(\sqrt{-1})) \leq 4u(k)$.

In each case, by the Milnor conjecture ([Voe03a, Voe03b]), we may choose a quadratic form $\phi \in I^m$ such that $e_m(\phi) = \alpha$. By our assumption on the $u(k)$, we may represent ϕ by an anisotropic form of dimension at most 2^n . By theorem 6, if $\dim(\phi) < 2^{m+1}$ it must be of the form $2^{m+1} - 2^i$ for some $1 \leq i \leq m+1$.

In case 1, we know that ϕ may be represented by an anisotropic form of dimension less than 2^n , which is in turn less than 2^m . However, $2^{m+1} - 2^i$ is either 0 or greater than or equal to 2^n . Therefore $\phi = 0$ and $\alpha = e_m(\phi) = 0$ as well.

Alternately, since k is C_n , every m -fold Pfister form is trivial. Since I^m is generated by m -fold Pfister forms, it must be trivial. Therefore, $\phi = 0$ and $\alpha = 0$.

In case 2, the dimension of an anisotropic representation of ϕ has dimension at most 2^n , and must be of the form $2^{n+1} - 2^n$. Therefore it is either 0 or of dimension exactly $2^{n+1} - 2^n = 2^n$. If we then choose a quadratic extension L/k such that ϕ_L is isotropic, then its anisotropic part will be a strictly smaller dimensional form in I^n , which again by theorem 6 immediately forces it to be 0. Therefore $\text{ind}(\alpha) | 2$.

In case 3, note that we may always find a tower of quadratic extensions such that each successive extension lowers the dimension of the anisotropic part of ϕ by at least 2. Since the anisotropic part has dimension at most 2^n , we may find a field extension L/k which is a tower of

$$\frac{2^n - (2^{m+1} - 2)}{2} = 2^{n-1} - 2^m + 1$$

many quadratic extensions such that ϕ_L has anisotropic part at most $2^{m+1} - 2$. By lemma 8, ϕ_L is split by an extension of degree dividing 2^m . Therefore, ϕ is split by an extension of degree

$$2^m 2^{2^{n-1} - (2^m - 1)} = 2^{2^{n-1} - 2^m + m + 1}.$$

□

Lemma 8. *Suppose $\psi \in I^m$ has dimension at most $2^{m+1} - 2^i$. Then there is an extension K/k of degree dividing 2^{m+1-i} which splits ψ .*

Proof. Choose any quadratic extension F/k such that ψ_F is isotropic. By theorem 6, its anisotropic part must have dimension at most $2^{m+1} - 2^{i+1}$ and so by induction it is split by an extension K/F of degree dividing $2^{m+1-(i+1)} = 2^{m-i}$. But this implies that $[K : k]$ divides 2^{m-i+1} and we are done. □

We now demonstrate that the bound described in conjecture 2 is sharp when k is the iterated power series field $F((t_1))((t_2)) \cdots ((t_n))$

with F algebraically closed. The same proof will work in general for a strictly Henselian valued field as described in [?].

Proposition 9. *Suppose k is as above. Then for any p , and for $m > 1$, if $\alpha \in H^m(k, \mu_p)$, then $\text{ind}(\alpha) | p^{n-m+1}$. Further, for $p = 2$, we may find α as above such that $\text{ind}(\alpha) = p^{n-m+1}$.*

Proof. Using [?], we may identify $G = \text{Gal}(k)$ with $(\widehat{\mathbb{Z}})^n$, and identify

$$K_1^M(k)/p = k^*/(k^*)^p = H^1(k, \mu_p) = \text{Hom}_{\text{cont}}((\widehat{\mathbb{Z}})^n, \mathbb{Z}/p) = (\mathbb{Z}/p)^n,$$

($K_i^M(k)$ denotes the Milnor K-theory of k), where the generators in $(\mathbb{Z}/p)^n$ correspond to taking p 'th roots of the uniformizers t_i . By the Bloch-Kato conjecture ([?]), we have an isomorphism $K_m^M(k)/p \cong H^m(k, \mu_p)$. Since $K_1^M(k)/p = \mathbb{Z}/p^n$, we may explicitly present $K_*^M(k)/p$ as the exterior algebra $\wedge^*(\mathbb{Z}/p)^n$ modulo the ideal generated by elements of the form $a \wedge (1 - a)$ for $a \in K_1^M(k)/p$. In the case a is a uniformizer, it is easy to check that $1 - a$ is a p 'th power, and therefore this ideal is in fact trivial. Hence we have an isomorphism $H^m(k, \mu_p) = \wedge^m(\mathbb{Z}/p)^n$. In particular, the symbols $(t_{i_1}, t_{i_2}, \dots, t_{i_m})$ with $i_1 < i_2 < \dots < i_m$ give a basis for $H^m(k, \mu_p)$.

We now prove the bound for the index given above by induction on n , beginning with the case $n = 1$, when the statement is classically known (see [?]). For general n , we begin by writing a general element $\alpha \in H^m(k, \mu_p)$ in the above basis. Suppose that the uniformizer t_n appears in some symbols in this expression. Let $L = k(t_n^{1/p})$. The restriction α_L is now a sum of symbols involving only the uniformizers t_1, \dots, t_{n-1} , and therefore may be considered as the restriction of an element $\beta \in H^m$ □

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