Math 2200 (Lieman), Fall 2002, Final(!) Review

The last two things we did in this class were to put together all of the geometry of curves we could (from derivatives, etc.) in order to graph curves, and to start to solve differential equations(!).

Graphing

We introduced one new idea in order to help us graph - that of the second derivative. The second derivative of a function f is just the derivative of the derivative. Sometimes this is denoted by $f''(x)$ and sometimes by $\frac{d^2f}{dx^2}$, but it's always just $(f'(x))'$. For example, if $f(x) = x^3 + 7x$, $f''(x) = 6x$; if $f(x) = \sin(x)$, $f''(x) = -\sin(x)$.

What's the second derivative good for? Well, suppose you're looking at a graph of $y = f(x)$, and you're at a minimum point $(x_0, f(x_0))$. In other words, near x_0, the graph has a higher value than $f(x_0)$. Well, that means that $f'(x_0) = 0$ since the tangent line is flat at $(x_0, f(x_0))$ but it *also* means that the first derivative is negative to the left of x_0 and positive to the right of x_0. In other words, it is *increasing* right at x_0. Well, if a function is increasing, its derivative is positive, so the derivative of the derivative (the second derivative!) is positive. Thus we conclude that a function $y = f(x)$ has a minimum at $(x_0, f(x_0))$ if two things are true: first, that $f'(x_0) = 0$, and second, that $f''(x_0)$ is positive. A similar reasoning works for maximum points as well.

In addition, if $f''(x) > 0$, then that means that $f'(x)$ is increasing - the graph is getting steeper and steeper (or if it is negative, less and less negative steep, if that makes sense - it's flattening out), so that means the graph of $y = f(x)$ is *concave up* (it's like a bowl, it would hold water). Similarly, if $f''(x) < 0$, the graph is concave down. Places where the graph changes concavity (from up to down, or vice versa) are called *inflection points:* those are places where both the first and second derivative vanish, and the second derivative changes sign on the two sides of that point.

We now have all the tools we need to draw graphs! Given a function $y = f(x)$, you graph it by:

First, think about what happens are you go off to $\pm\infty$. Remember that for polynomials, the largest term will always dominate when $|x|$ is large.

Second, compute the derivative and find the critical points (where $f'(x) = 0$). These are the possible max/min points.

Third, compute the second derivative. For each critical point, plug it into the second derivative. If the second derivative at a critical point is negative, that point corresponds to a (local, perhaps a global) maximum. If the second derivative at that point is positive, that critical point corresponds to a minimum (as in our discussion, above).

Fourth, figure out where the second derivative is positive or negative. That tells us where the graph is concave up or down.

Finally, draw the graph. It should be clear where it is increasing or decreasing, but if not, think about the first derivative - it is positive exactly when the graph is increasing, and negative when the graph is decreasing.

Differential Equations and the Antiderivative

Remember that if you know the derivative of a function, you know the SHAPE of a function - not where it is drawn. For example, x^2 and x^2+37 have the same shape (and the same derivative!), but aren't the same function. We introduce the notation

$$\int f(x)\,dx$$

for the *antiderivative* of $f(x)$ with despect to x. In other words, it is the general function $F(x)$ with the property that $F'(x)=f(x)$. Since lots of functions have the same derivative, we ALWAYS use a constant C to remind ourselves that the antiderivative includes all functions which differ by a constant.

For example, $\int(3x^2)\,dx = x^3+C$. Since differentiation has a bunch of rules (derivative of a sum is the sum of a derivative, etc.), so does antidifferentiation. Namely, if c is a constant, and $f(x)$ and $g(x)$ are functions, we have

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$
$$\int (cf(x))\,dx = c\int f(x)\,dx.$$

Our rules for differentiation translate into rules for antidifferentiation (sometimes called indefinite integration)

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}+C, \qquad n\neq -1$$
$$\int \frac{1}{x}\,dx = \ln(x)+C$$
$$\int e^x\,dx = e^x+C$$
$$\int \sin(x)\,dx = -\cos(x)+C$$
$$\int \cos(x)\,dx = \sin(x)+C$$

Using the general rules for antidifferentiation together with these formulas, we can do more complicated indefinite integrals. For example,

$$\int (x^2-2\sin(x))\,dx = \int x^2\,dx + \int -2\sin(x)\,dx =$$
$$= \int x^2\,dx - 2\int \sin(x)\,dx = \frac{x^3}{3}+2\cos(x)+C.$$

Finally, we can use antiderivatives to solve differential equations! If you know about the derivative of a function, remember, you don't know enough to solve for the function UNLESS you are given another specific piece of information (in order to figure out where to draw the function on the graph; the derivative tells you the shape, and the other piece of information tells you *where* to draw that shape). For example, if you know that $y' = x^2+4$, then that means $\frac{dy}{dx} = x^2+4$, and so

$dy = (x^2 + 4)\,dx$. Thus $\int dy = \int (x^2 + 4)\,dx$, and so, finally, $y = \frac{x^3}{3} + 4x + C$. You can't say any more - you can't say what y is exactly (no matter what value of C you plug in, it will have the right derivative). But if you know more, you can solve for the constant. For example, if $y' = x + 2$ AND $y(1) = 3$ (i.e. when $x = 1, y = 3$), then you can solve, as before, and you get $dy = (x + 2)\,dx$ and so $y = \frac{x^2}{2} + 2x + C$, BUT you can plug in $x = 1, y = 3$ and solve for C, to get $C = \frac{1}{2}$. Thus $y = \frac{x^2}{2} + 2x + \frac{1}{2}$.