Math 2200 (Lieman), Fall 2002, Solution to first exam

1. (15 points) Find the value of c that makes the following function continuous

$$f(x) = \begin{cases} x+5, & x \leq 2 \\ x^2 + c & x > 2 \end{cases}$$

In order for the function to be continuous everywhere, we must have

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$$

for all a. The only place where this might not work is at 2. So we have to make sure that

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x).$$

But

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + c) = 4 + c$$

and

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x + 5) = 7,$$

so for these two limits to be equal we must have $4 + c = 7$, or $c = 3$.

2. (15 points) This problem has two parts. The first part is worth ten points. The second part is worth five points. For both parts, the function $p(t)$ is a function that gives the location of a particle at time t, where

$$p(t) = 2t^3 + 7t + 1.$$

(a) Compute the derivative of $p(t)$ (usually denoted $\frac{dp}{dt}$ or $p'(t)$).

(b) What is the particle's velocity when $t = 3$?

The derivative of $2t^3 + 7t + 1$ is the sum of the derivatives of each of the terms. I.e.

$$p'(t) = (2t^3 + 7t + 1)' = (2t^3)' + (7t)' + (1)' = 2(3t^2) + 7 = 6t^2 + 7.$$

For part (b), note that the velocity is $p'(t)$, so the problem is asking for $p'(3)$. This is just $6(3)^2 + 7 = 61$.

3. (20 points) This problem has two parts, each worth ten points. For each part, you must evaluate a limit, or show it does not exist.

(a)

$$\lim_{x\to 4} \frac{x^2+2}{x-2}$$

(b)

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$$

The first limit is easy - if we plug in $x=4$, we see that everything makes sense, and we get $\frac{4^2+2}{4-2} = \frac{18}{2} = 9$.

The second limit is harder - if we try to plug in $x=4$, we get $\frac{0}{0}$, so we need to be clever. Since it is not a trig limit, we'll try to multiply by congugates to figure out what is really going on. We have

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{(\sqrt{x}-2)}{(x-4)} \frac{(\sqrt{x}+2)}{(\sqrt{x}+2)}$$
$$= \lim_{x\to 4} \frac{x-4}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{2+2} = \frac{1}{4}.$$

4. (20 points) This problem has two parts, each worth ten points. For each part, you must evaluate a limit, or show it does not exist.

(a)

$$\lim_{x\to 0} \frac{\sin(x)}{x}$$

(b)

$$\lim_{x\to 0} \frac{x^2}{1-\cos^2(2x)}$$

Part (a) is the key trig limit fact. Nothing more, nothing less. There's no other way to explain it. You simply must know that this limit is equal to 1.

For part (b), you can start by trying to plug in $x=0$, and you get $\frac{0}{0}$. So we know (since it is a trig limit), we have to try to turn it into something like part (a). First we'll try to get rid of the $\cos^2(2x)$, using $\sin^2(2x)+\cos^2(2x)=1$, then we'll try to get the same variable in the numerator and the denominator. Doing this, we see that

$$\lim_{x\to 0} \frac{x^2}{1-\cos^2(2x)} = \lim_{x\to 0} \frac{x^2}{\sin^2(2x)} = \lim_{x\to 0} \frac{4x^2}{4\sin^2(2x)}$$

and now we can write $t = 2x$, (so $t^2 = 4x^2$), to see that our limit is equal to

$$\lim_{t \to 0} \frac{t^2}{4 \sin^t(t)} = \lim_{t \to 0} \frac{1}{4} \frac{t}{\sin(t)} \frac{t}{\sin(t)} = \frac{1}{4}.$$

5. (20 points) This problem has two parts, each worth ten points. For each part, you must evaluate a limit, or show it does not exist.

(a)

$$\lim_{x \to 3} \frac{1}{(x-3)^2}$$

(b)

$$\lim_{x \to 3} \frac{1}{x-3}$$

For both parts, if you try to plug in $x = 3$, you get $\frac{1}{0}$. So we have to look at the left- and right-handed limits as $x \to 3$, to see if they agree (either $+\infty$ or $-\infty$).

For part (a), as we look at

$$\lim_{x \to 3^+} \frac{1}{(x-3)^2} \qquad \text{and} \qquad \lim_{x \to 3^-} \frac{1}{(x-3)^2},$$

we see it doesn't matter if x is bigger or smaller than 3, (and thus whether $x - 3$ is positive or negative) because we are squaring the denominator. So as $x \to 3$, $(x-3)^2$ goes to 0, but is always positive. So the term in the limit gets bigger and bigger, and is always positive, so each one sided limit goes to $+\infty$, and thus

$$\lim_{x \to 3} \frac{1}{(x-3)^2} = +\infty.$$

For part (b), we note that if $x > 3$, then $x - 3 > 0$, and so $\frac{1}{x-3}$ is always positive. As $x \to 3$, the denominator will go to 0, and so the whole thing (it's positive) goes off to $+\infty$. In other words,

$$\lim_{x \to 3^+} \frac{1}{x-3} = +\infty.$$

On the other hand, if $x < 3$ then $x - 3 < 0$, so $\frac{1}{x-3}$ will be negative. As before, as $x - 3$ goes to zero, this becomes a huge number, but it's negative, so

$$\lim_{x \to 3^-} \frac{1}{x-3} = -\infty.$$

Since the two one-sided limits don't agree, the limit in part (b) does not exist.

6. (10 points) This problem has two parts, each worth five points. Both parts concern the function

$$f(x) = 3x^2 + 6x + 7.$$

(a) What is the slope of the tangent line to $f(x)$? (You may use either the slope-predictor method or derivatives to answer this question.)

(b) Where is the tangent line to $f(x)$ horizontal?

The slope of the tangent line is the derivative. Period. So for part (a), we just have to compute $f'(x)$. Well,

$$f'(x) = (3x^2 + 6x + 7)' = (3x^2)' + (6x)' + (7)' = 3(2x) + 6 = 6x + 6$$

and that's the answer, $6x + 6$.

For part (b), we want to know where the tangent line is horizontal, i.e. has slope 0. That's true if $6x + 6 = 0$, or, in other words, where $x = -1$.