Math 2310H, Fall 2002, Solutions to first exam

1. (15 points) This problem has two parts. The first part is worth five points. The second part is worth ten points.

(a) Compute the integral

$$\int_0^{\pi/2} \sin^3(x)\cos(x)\,dx$$

(b) Compute the integral

$$\int_0^{\pi} \cos^4(x)\,dx$$

To compute the first integral, we make a simple substitution: $u = \sin(x), du = \cos(x)\,dx$. Then the integral transforms into $\int_0^1 u^3\,du = \frac{u^4}{4}]_0^1 = 1/4$.

To compute the second integral, we use the subtitution $\cos^2(x) = \frac{1+\cos(2x)}{2}$. Then we have

$$\int_0^{\pi} \left(\frac{1+\cos(2x)}{2}\right)^2 dx = \frac{1}{4}\int_0^{\pi} \left(1 + 2\cos(2x) + \cos^2(2x)\right)\,dx$$

$$= \frac{1}{4}\int_0^{\pi} \left(1 + 2\cos(2x) + \frac{1+\cos(4x)}{2}\right) dx = \frac{1}{4}\left(x + \sin(2x) + \frac{x}{2} + \frac{\sin(4x)}{8}\right)\Bigg]_0^{\pi}$$

which is $\frac{3\pi}{2}$ or something like that.

2. (15 points) This problem has two parts. The first part is worth five points. The second part is worth ten points.

(a) Compute the derivative

$$\frac{d}{dx}\int_3^{x} \sin(\sqrt{t+2})\,dt$$

(b) Compute the derivative

$$\frac{d}{dx}\int_3^{\cos x} \sin(\sqrt{t+2})\,dt$$

The first problem is a straightforward application of the FToC. We just use

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

and so the answer is $\sin(\sqrt{x+2})$.

The second problem is a bit more complicated. If we define

$$F(u) = \int_3^u \sin(\sqrt{t+2})\,dt,$$

then we calculated $F'(x)$ in part (a). But we want to know $[F(\cos(x))]'$. By the chain rule, this is $F'(\cos(x))(\cos(x))' = -\sin(\sqrt{\cos(x)+2})\sin(x)$.

3. (15 points) Find the volume obtained when the region under the curve $y = x^2$, $0 \le x \le 2$ is rotated around the $y-$axis.

You can do this by shells, or by disks. You're rotating around the vertical ($y-$) axis, so if you do shells, the volume is

$$\int_0^2 2\pi x x^2\,dx = 2\pi \frac{x^4}{4}\Big]_0^2 = 8\pi.$$

If you use disks, sliced horizontally, you have disks with a hole (i.e. washers) and the volume is

$$\int_0^4 \pi(2^2 - y)\,dy = \pi\left(4y - \frac{y^2}{2}\right)\Big]_0^4 = 8\pi.$$

4. (15 points) Find the arclength of the curve $y = x^{3/2}$, $0 \le x \le 3$.

This is a simple application of the arclength formula. $y' = \frac{3}{2}x^{1/2}$, so we get

$$\int_0^3 \sqrt{1 + \frac{9}{4}x}\,dx.$$

Now we can do the subtitution $u = 1 + \frac{9}{4}x$, so $du = \frac{9}{4}dx$ or $dx = \frac{4}{9}du$ and we get

$$\frac{4}{9}\int_1^{\frac{31}{4}} \sqrt{u}\,du = \frac{8}{27}u^{\frac{3}{2}}\Big]_1^{\frac{31}{4}} = \frac{4}{9}\left(\frac{31}{4}^{\frac{3}{2}} - 1\right).$$

5. (20 points) This problem has two parts. Each is worth ten points.

(a) Find the area between the curves $y = x^2$ and $y = x^3$, $0 \leq x \leq 2$.

(b) Write down an integral for the volume obtained when the area under the curve $y = \sin(x)$, $0 \leq x \leq \pi$ is rotated around the line $x = -2$.

Well, in the region $0 \leq x \leq 1$, $x^2 \geq x^3$, but in the region $1 \leq x \leq 2$, $x^3 \geq x^2$, so the answer is

$$\int_0^1 (x^2 - x^3)\, dx + \int_1^2 (x^3 - x^2)\, dx = \frac{3}{2}$$

or something like that.

For the second problem, we need to use shells. We're rotating around the vertical line $x = -2$, so the radius of a shell is $x - (-2) = x + 2$. The height is just $\sin(x)$, so we get

$$\int_0^\pi 2\pi(x+2)\sin(x)\, dx.$$

6. (20 points) This problem has three parts. The first part is worth ten points. The second and third parts are worth five points each.

(a) Write down an integral for the surface area generated when the curve $y = x^3$, $0 \leq x \leq 1$ is rotated around the line $x = 7$.

(b) Suppose $0 < a < b$ are constants (what they are is not relevant to the problem - this is just a technical detail so the rest of it works). Find a relationship between the constants A and B so that you can evaluate the integral of the arclength of $y = Ax^4 + \dfrac{B}{x^2}$, $a \leq x \leq b$ without having to do an integral involving square roots (i.e. find some way to make the square root in the formula go away!).

(c) Find the value of

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \sqrt{1 + k\frac{2}{n}}\ \frac{2}{n}.$$

The first problem is just a surface area integral. The radius of the surface is $7 - x$, and $y' = 3x^2$, so we get

$$\int_0^1 2\pi(7-x)\sqrt{1+9x^4}\,dx.$$

The second problem was a mess! When you differentiate, you get $y' = 4Ax^3 - 2Bx^{-3}$, and so

$$1 + (y')^2 = 1 + (16A^2x^6 - 16AB + 4B^2x^{-6}.$$

This will be a perfect square when that cross term $-16AB$ is equal to $-1/2$. Then adding the 1 will make it $+1/2$, and that's still a perfect square. So the answer is

$$16AB = \frac{1}{2} \Longrightarrow A = \frac{1}{32}B$$

(or, similarly, a relationship for B in terms of A).

The third problem is just the Riemann sum for the integral

$$\int_1^3 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big]_1^3 = \frac{2}{3}(3^{3/2} - 1).$$