Math 3000 (Lieman), Fall 2002, Solutions to second exam

1. (20 points) This problem has two parts, each worth ten points. For each part, determine whether the indicated subset is a vector subspace. You must justify your answer (as on every question on this test!) in order to get full credit.

(a) The set of vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ that satisfy $x_1+x_2+x_3 = 0$ and $x_1+x_3+x_4 = 0$.

Yes, this is a subspace. In fact, it's the nullspace of the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$ (i.e. it's just the solutions to $A\overrightarrow{\mathbf{x}} = 0$). Since every nullspace is a subspace, it's a subspace. Or, you can check quickly that it satisfies the three subspace properties.

(b) The set of vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ that satisfy $x_1+x_2+x_3 = 0$ and $x_1+x_3x_4 = 0$.

No, not a subspace. In fact, one point in this set is $\begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix}$. But the point $2\begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -2 \\ 8 \end{pmatrix}$ is not in this set, so it's not cloesd under scalar multiplication, so it's not a subspace. (It's also not closed under addition!)

2. (20 points) This problem has four parts, each worth five points.

(a) Are the two vectors $\overrightarrow{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\overrightarrow{\mathbf{v}}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ linearly independent?

Yes! The matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ has rank two, so the two vectors carve out a space of dimension two, so they are linearly independent.

(b) Do the two vectors $\overrightarrow{\mathbf{v}}_1$ and $\overrightarrow{\mathbf{v}}_2$ of part (a) span $\mathbb{R}^2$? Why or why not?

Yes, since they are linearly independent and there are two of them, they span $\mathbb{R}^2$ (so in fact, they are a basis of $\mathbb{R}^2$.

(c) Let A be the 3×3 matrix $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 6 \\ 5 & 2 & 9 \end{pmatrix}$. What is the rank of A? What is the dimension of the kernel of A?

Well, if we put A into row echelon form, we find that

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 6 \\ 5 & 2 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -3 \\ 0 & -8 & -6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -4 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

so A has rank 2, and so the dimension of the kernel of A is just $3 - 2 = 1$.

(d) Find a basis for the subspace (of $\mathbb{R}^4$) spanned by $\begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix}$.

Well, if we put the vectors into the columns of a matrix, and put that matrix into row echelon form, we get

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 3 & 1 & 4 & 2 \\ 2 & 4 & 6 & 4 \\ 1 & 1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & -2 & -1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & -2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We have pivots in columns 1, 2 and 4, so we select the first, second and fourth vector from our list, i.e. $\begin{pmatrix} 1 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix}$.

3. (20 points) Are the vectors $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$ a basis for $\mathbb{R}^4$?

Well, to be a basis for $\mathbb{R}^4$, we need four linearly independent vectors (which then automaticaly span $\mathbb{R}^4$ since there are four of them, and $\mathbb{R}^4$ has dimension 4).

Let's check if these are linearly independent. We make them the columns of a matrix, and then compute its rank.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 4 & 3 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & -4 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which has rank 3. So they aren't linearly independent, and hence they don't span $\mathbb{R}^4$. Since they don't span $\mathbb{R}^4$, they aren't a basis for $\mathbb{R}^4$.

4. (20 points) This problem has three parts. The first two parts are worth five points each. The third part is worth ten points.

(a) Suppose A is an invertible matrix such that $A^T = A^{-1}$. Show that $A\overrightarrow{\mathbf{x}} \cdot A\overrightarrow{\mathbf{y}} = \overrightarrow{\mathbf{x}} \cdot \overrightarrow{\mathbf{y}}$ for all pairs of vectors $\overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{y}}$.

We have

$$A\overrightarrow{\mathbf{x}} \cdot A\overrightarrow{\mathbf{y}} = \overrightarrow{\mathbf{x}} \cdot A^T A\overrightarrow{\mathbf{y}} = \overrightarrow{\mathbf{x}} \cdot A^{-1}A\overrightarrow{\mathbf{y}} = \overrightarrow{\mathbf{x}} \cdot \mathbb{I}\overrightarrow{\mathbf{y}} = \overrightarrow{\mathbf{x}} \cdot \overrightarrow{\mathbf{y}}.$$

(b) Find the transpose of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.

It's just $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

(c) Find the inverse of the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ -1 & 0 & 7 \end{pmatrix}$.

Well, we compute

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ -1 & 0 & 7 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 7 & 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 7 & -3 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 7 & -14 & 6 \\ 0 & 1 & 0 & -3 & 7 & -3 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{pmatrix}$$

so the inverse is just $\begin{pmatrix} 7 & -14 & 6 \\ -3 & 7 & -3 \\ 1 & -2 & 1 \end{pmatrix}$.

5. (20 points) This problem has three parts. The first two parts are worth five points each. The third part is worth ten points.

(a) Suppose A is an $n \times n$ matrix and $\overrightarrow{\mathbf{x}}$ and $\overrightarrow{\mathbf{y}}$ are nonzero vectors in $\mathbb{R}^n$ with

$$A\overrightarrow{\mathbf{x}} = 2\overrightarrow{\mathbf{x}}$$
$$A\overrightarrow{\mathbf{y}} = 3\overrightarrow{\mathbf{y}}.$$

Show that $\overrightarrow{\mathbf{x}}$ and $\overrightarrow{\mathbf{y}}$ are linearly independent.

Well, suppose

$$c_1\overrightarrow{\mathbf{x}} + c_2\overrightarrow{\mathbf{y}} = \overrightarrow{0}. \tag{1}$$

We want to show that $c_1 = c_2 = 0$. We know that $A(c_1 \overrightarrow{\mathbf{x}} + c_2 \overrightarrow{\mathbf{y}}) = A\overrightarrow{0} = \overrightarrow{0}$.

On the other hand,

$$(2.)\ \ \overrightarrow{0} = A(c_1 \overrightarrow{\mathbf{x}} + c_2 \overrightarrow{\mathbf{y}}) = A(c_1 \overrightarrow{\mathbf{x}}) + A(c_2 \overrightarrow{\mathbf{y}}) = c_1 A \overrightarrow{\mathbf{x}} + c_2 A \overrightarrow{\mathbf{y}} = 2c_1 \overrightarrow{\mathbf{x}} + 3c_2 \overrightarrow{\mathbf{y}}$$

If we take -2 times equation (1) and add it to equation (2), we get $c_2 \overrightarrow{\mathbf{y}} = 0$. Since $\overrightarrow{\mathbf{y}}$ is nonzero, $c_2 = 0$. But then equation (1) says $c_1 \overrightarrow{\mathbf{x}} = 0$, and since $\overrightarrow{\mathbf{x}}$ is nonzero, we must have $c_1 = 0$ as well, which is exactly what we wanted to show.

(b) Give an example of a linearly independent set of vectors in $\mathbb{R}^3$ that does not span $\mathbb{R}^3$.

Well, we want a linearly independent set, but which has fewer than 3 vectors. So $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is one such example. If you have three linearly independent vectors, you will span $\mathbb{R}^3$, so you just need to choose some set of fewer than 3 linearly independent vectors, and you're OK.