Math 3000 (Lieman), Fall 2002, Solutions to third exam

1. a) Find a vector $\vec{\mathbf{v}}$ so that the following set is orthogonal (in $\mathbb{R}^3$)

$$\left\{ \begin{pmatrix} \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \\ -\frac{4}{3\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \vec{\mathbf{v}} \right\}.$$

Well, we want a vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with the property that its dot product with each of the two vectors given above is equal to 0. The fact that its dot product with the second vector is equal to zero means that $a = b$, (so let's pick $a = b = 2$ for convenience). Then the fact that its dot product with the first vector is equal to zero determines c (in this case, $c = 1$). So our (one possible) answer is $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$. Note that you COULD do this by picking any third vector you want, and then using Gram-Schmidt. Since the first two are already (length 1 and) perpendicular, you'd end up changing only the third vector, which is exactly what you want.

b) Find a vector $\vec{\mathbf{v}}$ so that the list of vectors given in part a) is orthonormal.

Well, the two vectors given in part (a) already have length 1, so we only have to scale our third vector (divide by its length) to get it to have length 1. If we're starting with $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, we get $\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$.

c) Let L be the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ which acts by projecting a vector onto the x-axis. I.e. $L(\vec{\mathbf{x}})$ is the projection of $\vec{\mathbf{x}}$ onto the x-axis. Find a matrix A such that $A(\vec{\mathbf{x}}) = L(\vec{\mathbf{x}})$.

Well, if we can figure out $L(\vec{\mathbf{e}}_1)$ and $L(\vec{\mathbf{e}}_2)$ then our matrix will just be a matrix with first column equal to $L(\vec{\mathbf{e}}_1)$ and second column equal to $L(\vec{\mathbf{e}}_2)$. In fact, for any vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we have $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, so the answer is just $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

2. Use the Gram-Schmidt process to find an orthonormal set of vectors with the same span as the set

$$\left\{ \begin{pmatrix} 1 \\ 7 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 7 \\ 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 8 \\ 1 \\ 6 \end{pmatrix} \right\}.$$

I'll call these three vectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ and $\vec{\mathbf{v}}_3$. First we take $\vec{\mathbf{v}}_1$ and divide by its length.

$$\vec{\mathbf{w}}_1 = \frac{\vec{\mathbf{v}}_1}{\|\vec{\mathbf{v}}_1\|} = \frac{1}{\sqrt{1^2+7^2+1^2+7^2}}\begin{pmatrix}1\\7\\1\\7\end{pmatrix} = \frac{1}{10}\begin{pmatrix}1\\7\\1\\7\end{pmatrix}.$$

Now we have an intermediate second vector

$$\vec{\mathbf{u}}_2 = \vec{\mathbf{v}}_2 - (\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{w}}_1)\vec{\mathbf{w}}_1 = \begin{pmatrix}0\\7\\2\\7\end{pmatrix} - \left(\begin{pmatrix}0\\7\\2\\7\end{pmatrix} \cdot \frac{1}{10}\begin{pmatrix}1\\7\\1\\7\end{pmatrix}\right)\frac{1}{10}\begin{pmatrix}1\\7\\1\\7\end{pmatrix}$$

$$= \begin{pmatrix}0\\7\\2\\7\end{pmatrix} - \frac{1}{100}(0\cdot 1 + 7\cdot 7 + 2\cdot 1 + 7\cdot 7)\begin{pmatrix}1\\7\\1\\7\end{pmatrix} = \begin{pmatrix}0\\7\\2\\7\end{pmatrix} - \begin{pmatrix}1\\7\\1\\7\end{pmatrix} = \begin{pmatrix}-1\\0\\1\\0\end{pmatrix}.$$

Now $\vec{\mathbf{w}}_2 = \frac{\vec{\mathbf{u}}_2}{\|\vec{\mathbf{u}}_2\|} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$. To compute the third vector, we use a temporary third vector, $\vec{\mathbf{u}}_3 = \vec{\mathbf{v}}_3 - (\vec{\mathbf{v}}_3 \cdot \vec{\mathbf{w}}_1)\vec{\mathbf{w}}_1 - (\vec{\mathbf{v}}_3 \cdot \vec{\mathbf{w}}_2)\vec{\mathbf{w}}_2$, and then compute $\vec{\mathbf{w}}_3 = \frac{\vec{\mathbf{u}}_3}{\|\vec{\mathbf{u}}_3\|}$. If you don't make any mistakes, you should get $\vec{\mathbf{w}}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$.

3. Find the least squares solution $\vec{\mathbf{x}}$ to the inconsistent system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, where $A = \begin{pmatrix}1 & 1\\1 & 0\\0 & 1\end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix}3\\3\\3\end{pmatrix}$.

The least squares solution $\vec{\mathbf{x}}$ is just the vector that solves $A^T\vec{\mathbf{b}} = A^TA\vec{\mathbf{x}}$. If we multiply everything out, this is just

$$\begin{pmatrix}6\\6\end{pmatrix} = \begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}\vec{\mathbf{x}}$$

This is pretty easy to solve - the solution is just $\vec{\mathbf{x}} = \begin{pmatrix}2\\2\end{pmatrix}$. In other words, $x_1 = x_2 = 2$.

4. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by reflecting across the plane $-x_1 + x_2 + x_3 = 0$. Find the representation of L with respect to the standard basis. (Hint: you may first want to choose a basis of $\mathbb{R}^3$ for which it is easy to compute the action of L and then use a change of basis formula to get the final answer.)

We need to find a basis of $\mathbb{R}^3$ on which: 1) it is easy to figure out the action of L, and 2) the action of L is pretty simple. A great choice would be to choose a basis

$\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \overrightarrow{\mathbf{v}}_3$ where $\overrightarrow{\mathbf{v}}_1$ and $\overrightarrow{\mathbf{v}}_2$ are linearly independent vectors which lie in the plane $-x_1+x_2+x_3=0$ and $\overrightarrow{\mathbf{v}}_3$ is a normal vector to the plane. The $L(\overrightarrow{\mathbf{v}}_1)=\overrightarrow{\mathbf{v}}_1$, $L(\overrightarrow{\mathbf{v}}_2)=\overrightarrow{\mathbf{v}}_2$, and $L(\overrightarrow{\mathbf{v}}_3)=-\overrightarrow{\mathbf{v}}_3$.
We can make any choice we'd like for $\overrightarrow{\mathbf{v}}_1$ and $\overrightarrow{\mathbf{v}}_2$ as long as they are not on the same line, so how about $\overrightarrow{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, and $\overrightarrow{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. For $\overrightarrow{\mathbf{v}}_3$, we can use anything perpendicular to the plane, but the most natural choice is $\overrightarrow{\mathbf{v}}_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.
Then by the change of basis formula, the representation of L with respect to the standard basis is just

$$\underbrace{\begin{pmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}}_{P} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} P^{-1}.$$

5. Find the projection of the vector $\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$ onto the plane spanned by the two vectors $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

Well, if we let A be the matrix $\begin{pmatrix} 1 & 1 \\ -3 & 1 \\ 1 & 2 \end{pmatrix}$, and $\overrightarrow{\mathbf{b}}$ denote $\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$, then the solution is just $\overrightarrow{\mathbf{p}} = A\overrightarrow{\mathbf{x}}$, where $\overrightarrow{\mathbf{x}}$ solves $A^T\overrightarrow{\mathbf{b}} = A^TA\overrightarrow{\mathbf{x}}$. Rewriting this last equation (and computing out all the products, we have the system

$$\begin{pmatrix} -8 \\ 11 \end{pmatrix} = \begin{pmatrix} 11 & 0 \\ 0 & 6 \end{pmatrix} \overrightarrow{\mathbf{x}},$$

which has a solution $x_1 = -\frac{8}{11}$, $x_2 = \frac{11}{6}$. Thus our solution is just

$$\overrightarrow{\mathbf{p}} = \begin{pmatrix} 1 & 1 \\ -3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -\frac{8}{11} \\ \frac{11}{6} \end{pmatrix}.$$