The two faces of Blaschkean integral geometry

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Lecture 1: Introduction

General remarks

From our point of view, integral geometry is the study of the relationships among certain kinds of geometric integrals (G.-C. Rota called it “continuous combinatorics"). This study has two main aspects:

- The actual numerical relations between integrals of this type. As a result of the work of S. Alesker, this has become a problem in algebra (lectures 3, 4).
- Since the integrals are rather insensitive to the smoothness of the objects, one wants to understand the category of geometric objects to which these relations apply (lectures 2,3).

This subject is quite underdeveloped, yet not much sophistication is needed to appreciate it. As a result it has attracted the attention of many mathematicians over the years, usually unaware of each other. An extreme example is Chern’s 1966 paper, written in ignorance of Federer’s landmark work of 1959, which largely subsumes it; another is a very interesting 1978 paper of Milnor, unpublished until the issuing of his collected papers in 1994.

I would like to convince you that a basic vocabulary of the subject exists, sufficient to frame the most central questions in a precise way. The hope is that this will enable a more systematic development in the future.

What kinds of geometric integrals?

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- Buffon's needle problem: *When dropped at random on a plane ruled by parallel lines 1 unit apart, a needle of length* $L < 1$ *meets a ruling with probability* $\frac{2L}{\pi}$*.*
- Steiner's formula: *If* $K \subset \mathbb{R}^3$ *is a compact body, with either smooth or polyhedral boundary, then the volumes of the tubular neighborhoods* K_r *are given by the cubic polynomial*

$$|K_r| = |K| + P(K)r + H(K)r^2 + \frac{4\pi}{3}r^3$$

(figure 1a)

- Here $P(K) = |\partial K|$ and

$$H(K) = \begin{cases} \int_{\partial K} H & \text{if } K \text{ is smooth} \\ \sum_{\sigma \in 1-\text{skeleton}} |\sigma| \angle(K,\sigma) & \text{if } K \text{ is a polyhedron} \end{cases}$$

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Both of these facts are subsumed in Blaschke's **kinematic formulas**: if $K \in \mathcal{K}(\mathbb{R}^n) := \{\text{compact convex bodies in } \mathbb{R}^n\}$ then

$$\int_{\overline{SO(n)}} \mu_k(K \cap \bar{g}L)\, d\bar{g} = \begin{bmatrix} n+k \\ k \end{bmatrix} \sum_{i+j=n+k} \begin{bmatrix} n+k \\ i \end{bmatrix}^{-1} \mu_i(K)\mu_j(L), \tag{1}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} := \binom{n}{k} \frac{\omega_n}{\omega_k \omega_{n-k}},$$

$$\mu_i(K) := \int_{\mathrm{Gr}_i} |\pi_E(K)|\, dE, \quad \mu_i(K) = |K| \text{ if } \dim K = i;$$
$$d\bar{g}(\{\bar{g} : \bar{g}o \in S\}) = |S|.$$

- Taking $L = B_r$ yields Steiner's formula.
- The μ_i are **valuations**, i.e.

$$K, L, K \cup L \in \mathcal{K} \implies \mu_i(K \cup L) = \mu_i(K) + \mu_i(L) - \mu_i(K \cap L). \tag{2}$$

- Define $k_{SO(n)}(\mu_k) := \begin{bmatrix} n+k \\ k \end{bmatrix} \sum_{i+j=n+k} \begin{bmatrix} n+k \\ i \end{bmatrix}^{-1} \mu_i \otimes \mu_j$.

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Three remarkable facts

- Hadwiger (1957): The vector space $\mathrm{Val}^{SO(n)}$ of continuous $\overline{SO(n)}$-invariant valuations is spanned by the μ_i.
  Thus $k_{SO(n)} : \mathrm{Val}^{SO(n)} \to \mathrm{Val}^{SO(n)} \otimes \mathrm{Val}^{SO(n)}$, and this is a cocommutative, coassociative coproduct.
- Nijenhuis (1974): Putting

  $$\tau_i := \frac{i!\omega_i}{\pi^i}\mu_i = \frac{2^{i+1}}{\alpha_i}\mu_i,$$

  the kinematic formulas may be expressed

  $$\frac{2^{n+1}}{\alpha_n} k_{SO(n)}(\tau_k) = \sum_{i+j=n+k} \tau_i \otimes \tau_j, \tag{3}$$

  i.e. with this basis the structure constants of the rescaled coproduct are all equal to 1.
- Weyl's **"tube formula"** (1939): The μ_i extend in a natural way to compact smooth submanifolds $M \subset \mathbb{R}^n$, and the tube formula still holds. Moreover, the $\mu_i(M)$ are intrinsic invariants, i.e. they are integrals of scalar invariants of the Riemann curvature tensor, the **Lipschitz-Killing curvatures** of M.

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Federer's approach (1959)

H. Federer seems to have been the first to ask explicitly which subsets are subject to these laws.

Definition

For $A \subset \mathbb{R}^n$, reach(A) is the largest distance r_0 such that if $\mathrm{dist}(x, A) < r_0$ then there is a *unique* point $\pi_A(x) \in A$ with $|\pi_A(x) - x| = \mathrm{dist}(x, A) =: \delta_A(x)$. Thus reach$(A) = \infty$ iff A is convex. Say that $A \subset \mathbb{R}^n$ is **semiconvex** if it has locally positive reach. **(Figure 1b.)**

- V. Bangert, N. Kleinjohann (early 1980s): Semiconvexity is invariant under diffeomorphisms, so the notion of a semiconvex subset of a smooth manifold makes sense. If $a \in A$, A semiconvex, and the tangent cone Tan(A, a) has nonempty interior, then A is locally the diffeomorphic image of a piece of a convex body.

Theorem (Federer)

If A is semiconvex then there exist signed measures $\Phi_0^A, \dots, \Phi_n^A = \mathrm{vol} \llcorner A$ such that for bounded Borel sets $E \subset A$ and r small

$$\mathrm{vol}\left(\pi_A^{-1}(E) \cap \delta_A^{-1}[0, r]\right) = \sum_{i=0}^{n} \Phi_i^A(E) r^{n-i}. \tag{4}$$

If additionally A is compact, put $\mu_i(A) := c_i \Phi_i^A(A)$. Then the kinematic formulas (1) *hold for such pairs.*

The normal cycle

In the 1980s, P. Wintgen and M. Zähle independently observed that the Federer curvature measures are artifacts of the closed $(n-1)$-dimensional oriented Lipschitz submanifold

$$N(A) := \left\{ \left(\pi_A(x), \frac{x - \pi_A(x)}{|x - \pi_A(x)|} \right) : x \text{ near } A \right\} \subset \mathbb{UR}^n. \tag{5}$$

of unit normals to A: there are differential forms $\kappa_0, \ldots, \kappa_{n-1} \in \Omega^{n-1}(\mathbb{UR}^n)^{\overline{SO(n)}}$ such that

$$\Phi_i^A(E) = \int_{N(A)\cap\pi^{-1}(E)} \kappa_i, \quad i = 0, \ldots, n-1. \tag{6}$$

So strictly speaking one makes use only of $N(A)$ as *current*. This is the **normal cycle** of A. In fact the κ_i are simply appropriate multiples of the graded components of $i_{\partial_t} \exp^*(dx_1 \wedge \cdots \wedge dx_n)$ under the exponential map

$$\exp : \mathbb{R} \times \mathbb{UR}^n \to \mathbb{R}^n, \quad \exp(t, x, v) := x + tv.$$

If A, B intersect in a reasonable way (e.g. if they are polyhedra (Wintgen)) then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B) \tag{7}$$

The Legendre condition

Recall that the sphere bundle $\mathbb{UR}^n \simeq \mathbb{R}^n \times \mathbb{S}^{n-1}$ has a natural contact structure, with contact form

$$\alpha_{x,v} = \sum v_i dx_i$$

and horizontal distribution $Q := \alpha^\perp$. Then $d\alpha_{x,v}$ is a symplectic form on each vector space $Q_{x,v} \simeq v^\perp \oplus T_v\mathbb{S}^{n-1} \simeq v^\perp \oplus v^\perp$.

Now let $A \subset \mathbb{R}^n$ be a smooth compact body, and for $x \in \partial A$ put $L_x : T_x\partial A \to T_x\partial A$ for the Weingarten map. Put $n = n_A : \partial A \to \mathbb{S}^{n-1}$ for the Gauss map of outward pointing normals. Then

$$T_{x,n(x)}N(A) = \text{graph } L_x \subset Q_{x,n(x)}.$$

Furthermore this subspace is Lagrangian, which is equivalent to the fact that the Weingarten map is self-adjoint. Thus $\alpha|_{N(A)} = d\alpha|_{N(A)} = 0$, i.e. $N(A)$ is Legendrian.

Fact

The same is true if A is only semiconvex. In terms of currents this may expressed

$$N(A) \llcorner \alpha = 0 = N(A) \llcorner d\alpha. \tag{8}$$

Technical interlude: geometric integration theory

An oriented Lipschitz submanifold is a simple kind of **integral current**, as defined by Federer and Fleming in 1960. In order to push the ideas above further we need the full range of this language.

- A **current of dimension** k in a smooth Riemannian manifold M is a functional $T \in (\Omega_c^k(M))^*$ that is continuous with respect to C^∞ convergence of forms.
- The **boundary** of T is the $(k-1)$-dimensional current $\partial T(\phi) := T(d\phi)$.
- The **mass** of T is mass $T := \sup_{\|\phi\|_\infty \le 1} T\phi$, where $\| \phi \|_\infty := \sup\{\phi_x(v_1, \dots, v_k) : x \in M, v_i \in T_x M, |v_i| \le 1\}$ is the **comass**.
- The current T is **rectifiable** if

$$T = \sum_{i=1}^{\infty} f_{i*} [\![E_i]\!] \tag{9}$$

where the $E_i \subset \mathbb{R}^k$ are measurable and the $f_i : \mathbb{R}^k \to M$ are Lipschitz.

Definition

The abelian group $\mathbb{I}_k(M)$ of **integral currents of dimension** k consists of all k-dimensional locally rectifiable currents with boundary of locally finite mass.

It turns out that the boundary of an integral current is again integral. A smooth oriented submanifold V of dimension k, with smooth boundary, defines an element $[\![V]\!] \in \mathbb{I}_k(M)$ by integration. Stokes's theorem may then be stated: $\partial [\![V]\!] = [\![\partial V]\!]$.

Put $\|T\|_{U,\#} := \mathrm{mass}_U\, T + \mathrm{mass}_U\, \partial T$ (this notation is not standard!).

Theorem (Federer-Fleming (1960))

$$\mathbb{I}_k(M) = \bigcup_{C<\infty} \mathrm{clos}\left(\{T = \sum [\![V_i]\!] : V_1, V_2, \dots \textit{ smooth}, \ \|T\|_{U,\sharp} \le C(U) \textit{ for all } U \subset\subset M\}\right).$$

The closure here is with respect to flat convergence of currents, where

$$\|T\|_{U,\flat} := \sup_{\|\phi\|^\flat \le 1, \mathrm{spt}\,\phi \subset U} T\phi, \quad \|\phi\|^\flat := \sup\{\|\phi\|_\infty, \|d\phi\|_\infty\}.$$

Theorem (Slicing theorem)

Let $f : M \to N^n$ be a Lipschitz map, and $T \in \mathbb{I}_k(M)$, $k \ge n$. Then for a.e. $y \in N$ there exists $\langle T, f, y\rangle \in \mathbb{I}_{k-n}(M)$, supported in $f^{-1}(y)$, such that for every $\phi \in \Omega^{k-n}(M), \psi \in \Omega^n(N)$,

$$\int_T \phi \wedge f^*\psi = \int_N \left(\int_{\langle T,f,y\rangle} \phi\right) \psi(y), \textit{ and } \int_N \mathrm{mass}\langle T, f, y\rangle\, dy \le \mathrm{Lip}(f)\, \mathrm{mass}\, T \qquad (10)$$

If $T \in \mathbb{I}_k(M)$ then $\langle T, f, y\rangle \in \mathbb{I}_{k-n}(M)$ for a.e. $y \in N$, and

$$\partial\langle T, f, y\rangle = \langle \partial T, f, y\rangle.$$

A prototype for the Federer-Fleming theorem is

Theorem (BV compactness)

For $f \in L^1(U \subset \mathbb{R}^n)$*, put* $\|f\|_{BV} := \|f\|_1 + \text{mass}\, df$*. If* $U \subset\subset \mathbb{R}^n$ *and has smooth boundary then the inclusion* $BV(U) \to L^1(U)$ *is compact.*

This is something like the Arzela-Ascoli theorem: a BV function is “almost C^1" (though not necessarily C^0!), and the bound on mass df is something like equicontinuity.
The slicing theorem is a version of

Theorem (Coarea formula)

Let $f : M^m \to N^n$*,* $m \geq n$*, be a locally Lipschitz map between* C^1 *Riemannian manifolds, and* $E \subset M$*. Then*

$$\int_E J_n f(x)\, dx = \int_N |E \cap f^{-1}(y)|\, dy.$$

Here $J_n f(x)$ is the n-Jacobian of f, i.e. the euclidean norm of the map

$$\Lambda^n D_x f : \Lambda^n T_x M \to \Lambda^n T_{f(x)} N \simeq \mathbb{R}.$$

We will encounter the space $W^{1,BV}(U) := \{f : df \in BV(U)\}$. These functions are “almost C^2" — though we now introduce a more refined notion of this type:

Uniqueness for Lagrangian integral currents

Recall that if $f \in C^2(\mathbb{R}^n)$ then the manifold of unit outward normals to the subgraph

$$X_f := \{(x,t) : t \leq f(x)\}$$

is the image of the graph Γ_{df} of df under the map $\nu_f(x,\xi) := \left(x, f(x); \frac{(-\xi,1)}{\sqrt{1+|\xi|^2}}\right)$.

As a current, Γ_{df} may be characterized uniquely within $\mathbb{I}_n(T^*\mathbb{R}^n)$ as follows. Recall that such a graph is *Lagrangian*, i.e. that the restriction of the symplectic form vanishes.

Theorem (Fu, Jerrard)

Let $U \subset \mathbb{R}^n$ be open and $f \in W^{1,1}(U)$. Then there is at most one $T \in \mathbb{I}_n(T^*U)$ such that

1. $\partial T = 0$
2. $\text{mass}_{\pi^{-1}V}(T) < \infty$ *for every* $V \subset\subset U$
3. $T \llcorner \omega = 0$, *i.e.* T *is Lagrangian*
4. *for a.e.* $x \in U$,

$$\langle T, \pi, x\rangle = [\![d_x f]\!]. \tag{11}$$

If T exists and f is Lipschitz then

$$T \llcorner \alpha = T \llcorner \pi^* df$$

and $\nu_{f*} T \in \mathbb{I}_n(\mathbb{UR}^{n+1})$ is Legendrian.

Sketch of proof.

We show that if (1,2,3) all hold and $\langle T, \pi, x\rangle = 0$ for all $x \in \mathbb{R}^n$ then $T = 0$. As a model, if $V^n \subset T^*\mathbb{R}^n$ is a smooth Lagrangian submanifold and $\pi|_M$ is a submersion, then locally $V = N^* V' + dg$ for $V' = \pi(V)$ and some smooth function $g \in C^\infty(V')$. In particular, $\mathrm{vol}(V \cap \pi^{-1}(U)) = \infty$ for any $U \subset V'$ with positive measure. Likewise, if $\langle T, \pi, x\rangle \equiv 0$, let $\mathbb{R}^k \subset \mathbb{R}^n$ be a maximal coordinate subspace so that $\langle T, \pi_{\mathbb{R}^k}, y\rangle \neq 0$ for some non-null set of points $y \in \mathbb{R}^k$. Then each fiber of this slice over a point $x \in \mathbb{R}^n$ is supported in a finite union of parallel $(n-k)$-planes $P \subset T^*_x\mathbb{R}^n$ (i.e. the fibers look like perturbations of the conormal bundle of the image of the projection $\pi(T)$). Since the slices are themselves closed integral currents of dimension $n-k$, they must be integer multiples of the $[\![P]\!]$. Integrating over $\mathbb{R}^k$, and recalling the mass slicing inequality of (10), it follows that T cannot have locally finite mass. **(Figure 1c)** □

- If this T exists then we will denote it by Γ_{df}— the *differential cycle* of f— and refer to f as a **Monge-Ampère function**. The rationale is that the Hessian determinant of f then exists as a signed measure on $\mathbb{R}^n$, namely $\pi_*(\Gamma_{df} \llcorner d\xi_1 \wedge \cdots \wedge d\xi_n)$.
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Proposition

$MA(M)$ is closed under local uniform convergence with local mass bounds on differential cycles. If $f \in C^{1,1}(\mathbb{R}^n)$ then the mass of Γ_{df} is comparable to the sum of the L^1 norms of all minor determinants of the Hessian $D^2 f$. Therefore $\mathrm{MA}(\mathbb{R}^n)$ includes

- $W^{2,n}_{loc}(\mathbb{R}^n)$
- *all convex functions; in fact the differential current of a convex function f is simply the graph of the classical subdifferential $\{d\}f$, where $\{d\}_x f$ is the convex hull of all limits of legitimate differentials $d_{x'} f$ as $x' \to x$.*
- *all locally Lipschitz subanalytic functions, and all piecewise linear functions in particular.*

The proof: any such function is uniformly approximable by smooth functions, the graphs of whose differentials have locally bounded mass. Now apply Federer-Fleming compactness and the uniqueness theorem. In fact this is the only known method of producing MA functions:

Conjecture (Approximation Conjecture)

Every $f \in \mathrm{MA}$ is the local uniform limit of a sequence of C^∞ (or, equivalently, $C^{1,1}$) functions $f_1, f_2, \ldots$ such that for each $K \subset\subset \mathbb{R}^n$ there is $c(K) < \infty$ such that for all $I, J \subset \{1, \ldots, n\}, |I| = |J|$,

$$\int_K \left| \det \left(\frac{\partial^2 f_m}{\partial x_i \partial x_j} \right)_{i \in I, j \in J} \right| \leq c(K).$$

In particular, every MA function f is continuous.

Proposition

1. $\mathrm{MA}(\mathbb{R}) = W^{1,BV}(\mathbb{R}) = \mathrm{DC}(\mathbb{R})$, *the space of differences of convex functions.*
2. $\mathrm{DC}(\mathbb{R}^2) \subset \mathrm{MA}(\mathbb{R}^2) \subset C^0(\mathbb{R}^2)$.
3. *If* $n \geq 2$ *then* $\mathrm{MA}(\mathbb{R}^n)$ *is a proper subset of* $W^{1,BV}(\mathbb{R}^n)$*, and is not closed under addition.*
4. *If* $f, g \in \mathrm{MA}(\mathbb{R}^n)$ *then* $h_t(x) := f(x) + g(x+t) \in \mathrm{MA}(\mathbb{R}^n)$ *for a.e.* $t \in \mathbb{R}^n$.

- Jerrard has given examples to show that $f \in \mathrm{MA}(\mathbb{R}^2)$ may fail to be α-Hölder continuous for $\alpha > \frac{2}{n+1}$.
- Recall that the Hutchinson-Meier function

$$f(x) := x_1 \sin \log \log |x|^{-1}$$

belongs to $W^{2,n}(\mathbb{R}^n) \subset \mathrm{MA}(\mathbb{R}^n)$, while $\frac{\partial f}{\partial x_1}$ has unbounded variation along the x_1 axis.

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A dual version of the uniqueness theorem above characterizes the normal cycle of a general compact subset $X \subset \mathbb{R}^n$.

Theorem

Suppose $X \subset \mathbb{R}^n$ *is compact, and that for a.e.* $v \in \mathbb{S}^{n-1}$ *the function* $\chi_v(t) := \chi(X \cap h_v^{-1}(-\infty, t])$ *is well-defined and piecewise constant. Put* $\Delta\chi_v(t) := \chi_v(t+) - \chi_v(t-)$. *Then there is at most one* $T \in \mathbb{I}_{n-1}(S\mathbb{R}^n)$ *such that*

1. $\partial T = 0$
2. T *has compact support*
3. T *annihilates all multiples of the contact form* α *of* $S\mathbb{R}^n$
4. *if* $\phi \in C_0(\mathbb{R} \times \mathbb{S}^{n-1})$ *then*

$$T(\phi(x \cdot v, v)\, d\,\mathrm{vol}_{\mathbb{S}^{n-1}}) = \int_{\mathbb{S}^{n-1}} \sum_t \phi(t, v)\Delta\chi_v(t)\, dv \tag{12}$$

If this T exists, we denote it by $N(X)$ and call it the **normal cycle** of X. If X is semiconvex then it coincides with (integration over) the submanifold of outward unit normals constructed above. If X is subanalytic then it coincides with the image of Kashiwara's $CC(\mathbb{C}_X)$ under the antipodal map $(x, v) \mapsto (x, -v)$.

Definition

A compact set $X \subset \mathbb{R}^n$ is **geometric** if for a.e. $v \in S^{n-1}$ and every $t \in \mathbb{R}$, the set $X \cap h_v^{-1}(-\infty, t]$ is an absolute neighborhood retract, and $N(X)$ exists.

A theorem of West (1974) states that a compact metric ANR has the homotopy type of a finite complex.

Proposition

Any compact semiconvex set is geometric, as is any compact subanalytic set. An embedded curve then is geometric iff it has finite total curvature.

Proposition

If $X, Y, X \cap Y \subset \mathbb{R}^n$ are geometric sets, then so are

- $X \cup Y$, *with* $N(X \cup Y) = N(X) + N(Y) - N(X \cap Y)$.
- $X' := X \times (-\infty, 0] \subset \mathbb{R}^{n+1}$, *with* $N(X') := N(X) \divideontimes \llbracket e_{n+1} \rrbracket$ **(Figure 1d)**.
- $X^* := X \times \{0\} \subset \mathbb{R}^{n+1}$, *with* $N(X^*) := N(X) \divideontimes (\llbracket e_{n+1} \rrbracket - \llbracket -e_{n+1} \rrbracket)$

At this point one is tempted to define geometric subsets X of a smooth manifold M in terms of its images under local coordinate charts. Unfortunately, the theory is too shaky to fully support such a definition.
If $f \in C^{1,1}(\mathbb{R}^n)$ then $\nu_{f*}\Gamma_{df} = N(X_f)$ and $\gamma_* N(X_f) = \Gamma_{df}$ (ignoring the compactness condition...).
If $f \in \mathrm{MA}(\mathbb{R}^n)$ is Lipschitz then $\nu_{f*}\Gamma_{df}$ satisfies the first three conditions of the uniqueness theorem for normal cycles (if f is not Lipschitz then this pushforward operation is problematic).

Conjecture (Morse theory conjecture)

1. *Suppose* $f \in W^{1,1}(S^n)$ *Then* $f \in \mathrm{MA}(S^n)$ *iff* $X_f := \{x \in \mathbb{R}^{n+1} : |x| \le f\left(\frac{x}{|x|}\right)\}$ *is geometric. If* f *Lipschitz then*
$$\nu_{f*}\Gamma_{df} = N(X_f). \tag{13}$$
2. *If* X *is geometric and* $\phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *is a diffeomorphism, then* $\phi(X)$ *is geometric, with*
$$N(\phi(X)) = \tilde{\phi}_* N(X).$$
Here $\tilde{\phi} : \mathbb{UR}^n \to \mathbb{UR}^n$ *is the contactomorphism that covers* ϕ*.*

The matter is not completely clear even assuming the Approximation Conjecture for MA functions. The case $n = 1$ is easy. The case $n = 2$ is already interesting.

If X is compact and semiconvex then X_r is $C^{1,1}$ for small $r > 0$. If $X \subset \mathbb{R}^n$ is compact and subanalytic then $\mathrm{clos}(\mathbb{R}^n - X_r)$ is semiconvex for small r. In both cases $N(X) = \lim_{r \downarrow 0} N(X_r)$.

Conjecture (Nesting Conjecture)

Suppose $X \subset \mathbb{R}^{n+1}$ is compact, and that there exists a sequence of semiconvex sets $X_1 \supset X_2 \supset \ldots X$ such that $X = \bigcap_{i=1}^{\infty} X_i$ and mass $N(X_i) \leq C < \infty$ *for all i. Then X is geometric.*

Unlike the situation for Γ_{df}, it is certainly not the case that $N(X) = \lim_{i \to \infty} N(X_i)$ in general. Again the case $n = 2$ would be the place to start.