

Results and prospects in integral geometry

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Some classical facts

Theorem (Steiner)

If $A \in \mathcal{K}(\mathbb{R}^n) := \{\text{compact convex subsets of } \mathbb{R}^n\}$, or a smooth compact submanifold, then

$$\text{vol}(A_r) = \sum_{i=0}^n \omega_{n-i} \mu_i(A) r^{n-i}, \quad (r_0 \geq) r \geq 0.$$

Theorem (Blaschke kinematic formulas)

$$\int_{SO(n)} \mu_k(A \cap \bar{g}B) d\bar{g} = \sum_{i+j=n+k} c_{ij}^k \mu_i(A) \mu_j(B)$$

Theorem (Weyl)

If A is a smooth manifold then $\mu_i(A)$ is an integral of a scalar invariant of its curvature tensor.

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Theorem (Weyl)

If A is a smooth manifold then $\mu_i(A)$ is an integral of a scalar invariant of its curvature tensor.

Here μ_i is the *i*th intrinsic volume.

- μ_i is a valuation:

$$\mu_i(A \cup B) = \mu_i(A) + \mu_i(B) - \mu_i(A \cap B)$$

- If A is a smooth compact domain then

$$\begin{aligned}\mu_i(A) &= c_i \int_{\partial A} \sigma_{n-i-1}(k_1, \dots, k_{n-1}) \\ &= c_i \int_{\partial A} \text{trace} \left(\Lambda^{n-i-1} \Pi_A \right)\end{aligned}$$

Theorem (Hadwiger's theorem)

The vector space of continuous, translation-invariant valuations on the convex ring is spanned by μ_0, \dots, μ_n .

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Model questions

Analysis question

For what sets A, B are the kinematic formulas valid?

Algebra question

What are the coefficients c_{ij}^k ?

Geometry question

Why is Weyl's theorem true? What are its implications?

Chern: "Differential and integral geometry"

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- To summarize recent progress in the algebraic direction
- To describe how this progress might help us make progress in the geometric direction
- The analytic direction seems difficult, and I will only state a sharper (though not new) form of the question

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The normal cycle

Definition

If $A \in \mathcal{K}(\mathbb{R}^n)$:

$$N(A) = \{(x, v) \in \mathbb{R}^n \times S^{n-1} : x \in A, v, x - y \rangle \geq 0 \forall y \in A\}$$

If $P \subset M$ a smooth submanifold, $M =$ oriented Riemannian manifold:

$$N(P) = \{(x, v) \in SM : v \perp T_x P\}.$$

These objects are always **Legendrian cycles**, i.e. they annihilate the contact form of SM and have zero boundary.

Theorem

$N(A)$ makes sense for more general objects, and satisfies

$$\int_{N(A \cup B)} + \int_{N(A \cap B)} = \int_{N(A)} + \int_{N(B)}$$

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Smooth valuations and curvature measures

Definition

Let M^n = smooth oriented Riemannian manifold. A **smooth valuation** on M is a functional of the form

$$A \mapsto \int_A \theta + \int_{N(A)} \psi.$$

where $(\theta, \psi) \in \Omega^n(M) \times \Omega^{n-1}(SM)$.

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A **smooth curvature measure** assigns to each nice A the signed Radon measure on M given by

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Denote these vector spaces by

$$V(M), C(M)$$

- $\Omega^n(M) \times \Omega^{n-1}(SM) \rightarrow C(M) \rightarrow V(M)$.
- A smooth, $\Phi \in C \implies$

$$\Phi^A(E) = \int_{A \cap E} \theta + \int_{E \cap \partial A} f_\Phi(\Pi_A)$$

Theorem (Alesker)

$$\overline{(V^{\mathbb{R}^n}(\mathbb{R}^n)|_{\mathcal{K}})} = \text{Val}(\mathbb{R}^n)$$

:= continuous translation – invariant valuations

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Integral geometry in the language of valuations

Theorem (Existence of kinematic formulas)

If (M, G) is **isotropic** then $\dim V^G(M) \leq \dim C^G(M) < \infty$. There are linear maps

$$\begin{array}{ccc} C^G(M) & \xrightarrow{\tilde{k}_G} & C^G(M) \otimes C^G(M) \\ \downarrow & & \downarrow \\ V^G(M) & \xrightarrow{k_G} & V^G(M) \otimes V^G(M) \end{array}$$

such that

$$\begin{aligned} \tilde{k}_G(\varphi)(A, U; B, V) &= \int_G \varphi(A \cap gB, U \cap gV) dg, \\ k_G(\varphi)(K, L) &= \int_G \varphi(K \cap \bar{g}L) dg. \end{aligned}$$

- Proof: a formal construction based on the normal cycle.
- The **analysis question** restated: which sets A admit normal cycles? how is the normal cycle of A related to the geometry of A ?
- \tilde{k}_G, k_G are **cocommutative, coassociative coproducts**

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List of simply connected isotropic spaces

Among simply connected spaces, kinematic formulas exist in precisely the following cases:

- \mathbb{R} space forms: $S^n, (\mathbb{R}^n, SO(n)), H^n$
(classical; $\ker(C \rightarrow V) = 0 \implies k_{SO(n)} = \tilde{k}_{SO(n)}$)
- \mathbb{C} space forms: $\mathbb{C}P^n, (\mathbb{C}^n, U(n)), \mathbb{C}H^n$
(Bernig-Fu-Solanes; more about this later, also Gil's talk)
- \mathbb{H} space forms: $\mathbb{H}P^n, (\mathbb{H}^n, Sp(n) \times Sp(1)), \mathbb{H}H^n$
(partial results by Bernig, only for V)
- \mathbb{O} planes: $\mathbb{O}P^2, (\mathbb{O}^2, Spin(9)), \mathbb{O}H^2$
(nothing is known)
- ... together with the euclidean spaces on the next slide...

Theorem (Transfer principle)

Across each of these triples, the coalgebras $(C^G(M), \tilde{k}_G)$ are canonically isomorphic.

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More isotropic euclidean spaces

Bernig has calculated $k_G : V^G \rightarrow V^G \otimes V^G$ for $(M, G) =$

- $(\mathbb{C}^n, SU(n))$
- (\mathbb{R}^7, G_2)
- $(\mathbb{C}^4, Spin(7))$

and has partial results for

- $(\mathbb{H}^n, Sp(n))$
- $(\mathbb{H}^n, Sp(n) \times U(1))$

The algebraic breakthrough: Alesker product

Alesker-McMullen: We may take as models the discontinuous valuations

$$\mu_X : \mathbf{A} \mapsto \chi(\mathbf{A} \cap X).$$

Theorem (Alesker)

- *The product*

$$\mu_X \cdot \mu_Y := \mu_{X \cap Y}$$

extends to a commutative product on $V(M)$ with

$$\chi = 1$$

- *(Poincaré duality) If M is compact then the pairing on $V(M)$ given by*

$$(\phi, \psi) = (\phi \cdot \psi)(M)$$

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Suppose (M, G) is isotropic. Put

$$pd : V^G(M) \rightarrow V^{G^*}(M)$$

for the isomorphism induced by the pairing $(,)$.

Theorem (Ftaig)

The following diagram commutes:

$$\begin{array}{ccc} V^G(M) & \xrightarrow{k_G} & V^G(M) \otimes V^G(M) \\ pd \downarrow & & pd \otimes pd \downarrow \\ V^{G^*}(M) & \xrightarrow{m_G^*} & V^{G^*}(M) \otimes V^{G^*}(M) \end{array} \quad (1)$$

In other words

$$k_G(\chi) = pd^{-1}, \quad (\mu \otimes \chi) \cdot k_G(\phi) = k_G(\mu \cdot \phi)$$

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Analytic continuation

- In the $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ cases we have such a diagram for each curvature $\lambda \in (-\infty, \infty)$.
- By the Transfer Principle, the kinematic operators

$$\tilde{k}_G : C^G(M) \rightarrow C^G(M) \otimes C^G(M)$$

on curvature measures are independent of λ .

- In this common model, the Alesker product and Poincaré duality depend analytically on λ .

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Theorem (Enhanced Hadwiger theorem)

$$V_{\mathbb{R},0}^n \simeq \mathbb{R}[t]/(t^{n+1})$$

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If M^n is a Riemannian manifold then there is a canonical injection

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In particular, t is a well-defined element of each

$$V_{\mathbb{R},\lambda}^n := \{\text{invariant valuations on the space form of curvature } \lambda\}$$

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Previous work on the integral geometry of the M_λ^n : Shifrin kinematic formulas for pairs of complex analytic submanifolds; Gray's tube formulas.

Theorem (Gray 1984)

If $P \subset M_\lambda^n$ is a smooth complex analytic submanifold then

$$\int_{(N(P) \cap \pi^{-1}E) \times [0,r]} \exp^*(d \operatorname{vol}_{M_\lambda^n}) = \sum_{i=0}^n \lambda^{-i} \sin^{2i}(r\sqrt{\lambda}) \cos^{2n-2i}(r\sqrt{\lambda}) \int_E F_i(\Omega_P).$$

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The valuations approach (Bernig-Fu-Solanes)

Facts

- $(C^{U(n)}, \tilde{k})$ is independent of λ .
- $\ker(C^{U(n)} \rightarrow V_{\mathbb{C}, \lambda}^n) \neq 0$.
- $C^{U(n)}$ is a module over each $V_{\mathbb{C}, \lambda}^n$.
- All products and coproducts are compatible.

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For \mathbb{C}^n , $\mathbb{C}P^n$ or $\mathbb{C}H^n$ we put

$$s := \int_{\overline{\text{Gr}}_{n-1}^{\mathbb{C}}} \chi(\cdot \cap \bar{P}) d\bar{P}.$$

Theorem

- $V_{\mathbb{C},\lambda}^n$ is generated by s, t .
- $V_{\mathbb{C},0}^n \simeq \mathbb{R}[s, t]/(f_{n+1}, f_{n+2})$, where $\sum f_i = \log(1 + s + t)$.

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Structure of $V_{\mathbb{C},\lambda}^n$

Theorem (BFS)

Each of the following determines an isomorphism $V_{\mathbb{C},\lambda}^n \rightarrow V_{\mathbb{C},0}^n$:

- $t \mapsto \frac{t}{\sqrt{1-\lambda s}}, \quad s \mapsto s$
- $t \mapsto \frac{t}{\sqrt{1-\lambda \frac{t^2}{4}}}, \quad s \mapsto s$

Conjecture

Let s, t, λ be variables of formal degrees 2, 1, -2 respectively. Let

$$\sum \bar{f}_i = \log(1 + s + t + \sum_{k \geq 1} \left[\binom{4k+1}{k+1} - 9 \binom{4k+1}{k-1} \right] \lambda^k).$$

Then

$$V_{\mathbb{C},\lambda}^n \simeq \mathbb{R}[s, t] / (\bar{f}_{n+1}, \bar{f}_{n+2}, t^{2n+1}, st^{2n-1}, \dots, s^n t).$$

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Geometric consequences

Theorem (BFS)

There exist $\mu_i^\lambda \in V_{\mathbb{C},\lambda}^n$ such that for any nice compact subset $A \subset M_\lambda^n$ and small $r > 0$

$$\text{vol}_{2n}(A_r) = \sum_{i=0}^{2n} \omega_i \mu_{2n-i}^\lambda(A) \lambda^{-\frac{i}{2}} \sin^i(r\sqrt{\lambda}) \cos^{2n-i}(r\sqrt{\lambda}).$$

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If $P \subset M_\lambda^n$ is a totally real submanifold then

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In particular, the coefficients of the tube formula for P are Riemannian invariants.

This is a **global** formula, i.e. does not calculate $\exp^*(d \text{ vol})$.
Nevertheless, it implies Gray's local formula for subvarieties because

Fact

There is a subspace $Q \subset C^{U(n)}$, $\text{codim } Q = n + 1$, such that

$$\Phi^P = 0$$

whenever $P \subset M_\lambda^n$ is a subvariety and $\Phi \in Q$. Moreover

$$C^{U(n)} / Q \simeq \langle \mu_0^\lambda, \mu_2^\lambda, \dots, \mu_{2n}^\lambda \rangle$$

BFS also have worked out

- a complicated form for the kinematic operator

$$\tilde{k} : C^{U(n)} \rightarrow C^{U(n)} \otimes C^{U(n)}$$

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The geometric question: how far does the Weyl Principle go?

The Weyl Principle

Invariant valuations and curvature measures, when restricted to subspaces X , are determined by the intrinsic geometry of X .

Some instances:

- Weyl's tube formula for submanifolds of \mathbb{R}^n
- Gray's tube formula for subvarieties of M_λ^n
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Given any Riemannian manifold M , characterize t directly as an element of $V(M)$. Give a construction of t that also works if M is a Finsler manifold.

Conjecture

Let $P \subset M_{\mathbb{C},\lambda}^n$ be a smooth submanifold. Then the restriction of any $\Phi \in C^{U(n)}$ or $\mu \in V_{\mathbb{C},\lambda}^n$ can be computed from the induced Riemannian metric on P and the restriction of the Kähler form to P .

Understanding the integral geometry of the complex space forms will help us to try ideas and to calculate examples.

It also promises to uncover interesting new Riemannian phenomena:

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Angular curvature measures

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Let $\Phi \in C(M)$. For any differentiable polyhedron $P \subset M$

$$\Phi^P = \sum_{\text{faces } F^k} \delta_{P,F} \times \text{vol}_k|_F.$$

Definition

$\Phi \in C(M)$ is **angular** if

$$H_x^F = 0 \implies \delta_{P,F}(x) = c_{T_x F} \angle(P, F)$$

For example, suppose $\Phi \in C(\mathbb{R}^n)$ is translation-invariant and of degree k . Then $\Phi \in \mathcal{A}(\mathbb{R}^n)$ iff

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Denote the vector space of angular curvature measures on M by $\mathcal{A}(M)$.

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$$t \cdot (\mathcal{A}^{U(n)}(M_\lambda^n)) \subset \mathcal{A}(M_\lambda^n)$$

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