Due Wednesday, December 16.
Rules: No discussions with other students or faculty besides me. No sources besides your notes and the text.

Problem 1. (a) *Prove that $L^p(\mathbb{R}, m)$ is separable for $1 \leq p < \infty$. (Hint: consider the set of all rational linear combinations of characteristic functions of intervals with rational endpoints.)*

(b) *Show that there exists an uncountable set $S \subset l^\infty$ such that if $x, y \in S$, $x \neq y$, then $\| x - y \|_\infty \geq 1$.*

(c) *Prove that l^∞ is not separable. (Hint: if D is a dense subset and S is as in part (a), consider the sets $D_x := B(\frac{1}{2}, x) \cap D, x \in S$.)*

(d) *Let $E \subset \mathbb{R}$ be a measurable set with $m(E) > 0$. Construct an injective map $T : l^\infty \to L^\infty(E, m)$ such that $\| Tx \|_\infty = \| x \|_\infty$ for all $x \in l^\infty$.*

(e) *Conclude that $L^\infty(E, m)$ is not separable for E as in* (d).

Solution. a) It is enough to do this for $\mathbb{R}$-valued functions. We show that the countable set

$$S := \left\{ \sum_{i=1}^{N} q_i 1_{[r_i, s_i]} : q_i, r_i, s_i \in \mathbb{Q}, i = 1, \dots N \right\}$$

is dense. By Prop. 6.7 of Folland, the simple functions are dense in L^p. Thus it is enough to show that any simple function $\in L^p$ can be approximated arbitrarily closely by elements of S.

Let $\phi = \sum_{i=1}^N a_i 1_{E_i}$ be such a function. By the dominated convergence theorem, the functions $\phi \cdot 1_{[-n,n]} \to \phi$ in L^p as $n \to \infty$. Thus we may assume that all $E_i \subset [-n_0, n_0]$ for some $n_0 \in \mathbb{N}$. By Prop. 1.20 of Folland, for each $m \in \mathbb{N}$ there is a set $J_{i,m}$ that is a finite union of intervals such that $m(E_i \triangle J_{i,m}) < \frac{1}{m}$. Clearly the intervals comprising the $J_{i,m}$ may be taken to have rational endpoints and $\subset [-n_0, n_0]$.

Now let $\mathbb{Q} \ni q_{i,m} \to a_i$ as $m \to \infty$. In particular $|a_i|, |q_{i,m}| \leq C$ for all i, m and some $C < \infty$. Then $\phi_m := \sum_{i=1}^N q_{i,m} 1_{J_{i,m}} \to \phi$ a.e. as $i \to \infty$, and clearly $|\phi_m - \phi|^p \leq (N+1)^p C^p \cdot 1_{[-n_0, n_0]}$. Therefore the dominated convergence theorem applies to show that $\| \phi_m - \phi \|_p \to 0$ as $m \to \infty$.

b) Take $S := \{1_E : E \subset \mathbb{N}\}$. If $E \neq F$ and then

$$\| 1_E - 1_F \|_\infty = \| 1_{E \triangle F} \|_\infty = 1.$$

Since S is in one-to-one correspondence with the power set of $\mathbb{N}$, which is uncountable, S is itself uncountable.

c) Suppose $D \subset l^\infty$ is dense. Then $D_x := D \cap B(\frac{1}{2}, x) \neq \emptyset$ and for each $x \in S$, and by the triangle inequality the D_x are pairwise disjoint. So $D \supset \bigcup_{x \in S} D_x$ is uncountable.

d) Let $E' \subset E$ be a subset of finite positive measure. Put $f(x) := m(E' \cap (-\infty, x])$. By the intermediate value theorem there is t_0 such that $E' \supset$

$E_1 := E' \cap (-\infty, x])$ has $m(E_1) = \frac{1}{2}m(E')$. Applying this process to $E' \backslash E_1$, we obtain $E \backslash E_1 \supset E_2$ with $m(E_2) > 0, m(E \backslash (E_1 \cup E_2)) > 0$. Proceeding inductively, this gives a sequence of disjoint subsets $E_1, E_2, \dots \subset E$, all with positive measure (in fact $m(E_i) = 2^{-i}m(E')$).

For $x \in l^\infty$ we put $Tx := \sum x_i 1_{E_i}$. Clearly T is linear, with $\| Tx \|_\infty = \| x \|_\infty$.

e) Since $\| Tx - Ty \|_\infty = \| T(x-y) \|_\infty = \| x - y \|_\infty = 1$ for $x, y \in S$, we may apply the argument of c).

Problem 2. *Construct an example of a function $f \in L^2(\mathbb{R}, m)$ such that* $\operatorname{ess\,sup}_U f = \infty$ *for every open set $U \subset \mathbb{R}$.*

Solution. Define $f(x) := \begin{cases} x^{-\frac{1}{3}}, & 1 > x > 0 \\ 0 & \text{otherwise} \end{cases}$. Then $f \in L^2$ and is essentially unbounded on every neighborhood of 0. Let $q_1, q_2, \dots$ be an enumeration of $\mathbb{Q}$. Then $f_i(x) := 2^{-i} f(x - q_i) \in L^2$ and is essentially unbounded on every neighborhood of q_i, with $\| f_i \|_2 = 2^{-i} \| f \|_2$. Therefore the absolutely convergent sum $g := \sum_{i=1}^\infty f_i \in L^2$. Since $g \geq f_i$ for every i, it is essentially unbounded on every neighborhood of $q_i, i = 1, 2, \dots$. Since any open set $U \ni$ some q_i, this is what we want.

Problem 3. *Suppose $g : [0,1] \to \mathbb{R}$ is absolutely continuous.*

(a) *Prove that if $E \subset [0,1]$ with $m(E) = 0$ then $m(g(E)) = 0$.*
(b) *Put $F := \{x \in [0,1] : g'(x) = 0\}$. Prove that $m(g(F)) = 0$. (Use (a) and the Vitali covering theorem.)*
(c) *Conclude from (a) and (b) that for a.e. $y \in \mathbb{R}$, the function g is differentiable at every point $x \in g^{-1}(y)$, with $g'(x) \neq 0$.*
(d) *Prove that for y as in (c) the preimage $g^{-1}(y)$ is finite.*
(e) *Put $N(y)$ for the number of points $\in g^{-1}(y)$. Prove that*

$$\int_{[0,1]} |g'| = \int N(y)\, dm(y).$$

Solution. a) Let $\epsilon > 0$ be given, and following the definition of absolute continuity let $\delta > 0$ be such that

$$0 \leq a_1 < b_1 \leq a_2 < \dots b_N \leq 1, \ \sum^N (b_i - a_i) < \delta \implies \sum_N |g(b_i) - g(a_i)| < \epsilon.$$

Let $\bigcup^\infty (\alpha_i, \beta_i) \supset E$, with $\sum^\infty (\beta_i - \alpha_i) < \delta$. Then each $g([\alpha_i, \beta_i])$ is an interval $[c_i := \min_{[\alpha_i, \beta_i]} g, d_i := \max_{[\alpha_i, \beta_i]} g]$, and $\bigcup^\infty [c_i, d_i] \supset g(E)$. Thus it is enough to show that $\sum^\infty (d_i - c_i) \leq \epsilon$, and to show this it is enough in turn to show that $\sum^N (d_i - c_i) < \epsilon$ for every $N \in \mathbb{N}$.

Let $\alpha_i \leq a_i < b_i \leq \beta_i$ such that $\{g(a_i), g(b_i)\} = \{c_i, d_i\}$ (possible since g is continuous). Then $\sum^N (b_i - a_i) < \sum^N (\beta_i - \alpha_i) < \delta$, so $\sum^N (d_i - c_i) = \sum^N |g(b_i) - g(a_i)| < \epsilon$ as claimed.

b) Let $\epsilon > 0$. By the definition of F, for each $x \in F$ there is $\delta_x > 0$ such that $g([x-h, x+h]) \subset [g(x) - \epsilon h, g(x) + \epsilon h]$ whenever $0 < h < \delta_x$. In particular,

$$m(g([x-h, x+h])) < \epsilon m([x-h, x+h])$$

The set F and the family of such intervals $[x-h, x+h]$ satisfy the hypotheses of the Vitali covering theorem, yielding the existence of a countable family of disjoint intervals $I_j = [a_j, b_j] \subset [0,1]$, $j = 1, 2 \ldots$, such that $m(F \setminus \bigcup_{j=1}^{\infty} I_j) = 0$ and

$$m(g(I_j)) < \epsilon m(I_j), \quad j = 1, 2, \ldots.$$

Since all $I_j \subset [0,1]$,

$$\sum^{\infty} m(I_j) = m(\bigcup^{\infty} I_j) \leq m([0,1]) = 1.$$

Then using part a)

$$m(g(F)) \leq m(g(F \setminus \bigcup I_j)) + \sum_j m(g(I_j)) \leq \sum \epsilon m(I_j) < \epsilon$$

Since this holds for every $\epsilon > 0$, we conclude that $m(g(F)) = 0$.

c) Put $E \subset [0,1]$ to be the set of points at which g is not differentiable. Then $m(E) = 0$, so by parts a) and b)

$$m(g(E \cup F)) = m(g(E) \cup g(F)) \leq m(g(E)) + m(g(F)) = 0.$$

By definition any $y \notin g(E \cup F)$ has the properties required.

d) If $g^{-1}(y)$ is infinite then by compactness there is a convergent sequence $g^{-1}(y) \ni x_1, x_2, \cdots \to x_0 \in [0,1]$. Then $g(x_0) = y$ by continuity of g, so $g'(x_0)$ exists, and

$$g'(x_0) = \lim_{n\to\infty} \frac{g(x_n) - g(x_0)}{x_n - x_0} = \lim_{n\to\infty} \frac{y - y}{x_n - x_0} = 0.$$

This is a contradiction.

e) For continuous functions $f : [a,b] \to \mathbb{R}$, put $\#_f(y)$ to be the number (possibly ∞) of points in $f^{-1}(y)$. If $a < c < b$ then clearly

$$\#_f(y) = \#_{f|_{[a,c]}}(y) + \#_{f|_{[c,b]}}(y) \tag{1}$$

for $y \neq f(c)$, and if $f([a,b]) = [\alpha, \beta]$ then

$$\#_f \geq 1_{[\alpha,\beta]} \tag{2}$$

with equality iff f is monotone. We want to show that

$$\int \#_g(y)\, dm(y) = \int |g'|$$

for absolutely continuous g.

For a given partition $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_N = 1\}$, put $\parallel \mathcal{P} \parallel := \max |t_i - t_{i-1}|$. Put $g_{\mathcal{P}}$ to be the piecewise linear function such that $g_{\mathcal{P}}(t_i) =$

$g(t_i)$ for all $t_i \in \mathcal{P}$, and is linear between each t_{i-1}, t_i. By the intermediate value theorem,

$$\#_{g_{\mathcal{P}}} \le \#_g. \tag{3}$$

Since $g_{\mathcal{P}} = (g_{\mathcal{Q}})_{\mathcal{P}}$ when $\mathcal{Q} \supset \mathcal{P}$ it follows that

$$\#_{g_{\mathcal{P}}} \le \#_{g_{\mathcal{Q}}} \tag{4}$$

in this case.

Recall that

$$\int |g'| = T(g) = \sup_{\mathcal{P}} T(g, \mathcal{P})$$

where

$$T(g, \mathcal{P}) = \sum_{i=1}^{N} |g(t_i) - g(t_{i-1})| = \int \#_{g_{\mathcal{P}}}(y)\, dm(y)$$

The second equality follows from (1) and the equality case of (4).

Let $\mathcal{P}_i, i = 1, 2, \dots$ be partitions such that $T(g, \mathcal{P}_i) \to T(g)$. Since $T(g, \mathcal{P}) \ge T(g, \mathcal{Q})$ whenever $\mathcal{P} \supset \mathcal{Q}$, we may assume that $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \dots$ and that $\| \mathcal{P}_i \| \to 0$.

Suppose y is as in c). Then there is $\delta > 0$ such that if $-\delta < h' < 0 < h < \delta$ then $g(x)$ lies between $g(x+h)$ and $g(x+h')$ for every $x \in g^{-1}(y)$. Using the intermediate value theorem, it follows that if $\| \mathcal{P} \| < \delta$ then $\#_{g_{\mathcal{P}}}(y) \ge \#_g(y)$. With (3) this implies that $\#_{g_{\mathcal{P}}}(y) = \#_g(y)$ in this case.

Now (4) and the result of c) imply that $\#_{g_{\mathcal{P}_i}} \uparrow \#g$ a.e. Applying the monotone convergence theorem to this sequence gives the result we want.

Problem 4. *Suppose* F, G *are absolutely continuous on* $[0,1]$. *Prove that* FG *is absolutely continuous.*

Solution. Since F, G are continuous there is $C < \infty$ such that $|F|, |G| \le C$. Given $\epsilon > 0$ let $\delta > 0$ be small enough that if $0 \le a_1 < b_1 \le a_2 < \dots b_N \le 1, \sum^N (b_i - a_i) < \delta$ then $\sum^N |F(b_i) - F(a_i)|, \sum^N |G(b_i) - G(a_i)| < \frac{\epsilon}{2C}$. Then for such a_i, b_i

$$\begin{aligned}
\sum^N |F(b_i)G(b_i) - F(a_i)G(a_i)| &= \sum^N |F(b_i)G(b_i) - F(b_i)G(a_i)| + \sum^N |F(b_i)G(a_i) - F(a_i)G(a_i)| \\
&= \sum^N |F(b_i)||G(b_i) - G(a_i)| + \sum^N |F(b_i) - F(a_i)||G(a_i)| \\
&\le C\left(\sum^N |G(b_i) - G(a_i)| + \sum^N |F(b_i) - F(a_i)|\right) \\
&< C\left(\frac{\epsilon}{2C} + \frac{\epsilon}{2C}\right) = \epsilon.
\end{aligned}$$

Problem 5. *Put*

$$g(x) := \begin{cases} \frac{3}{4}(1 - x^2), & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

and for $t \neq 0$

$$g_t(x) := \frac{1}{t} g(\frac{x}{t}).$$

Let $f \in L^1(\mathbb{R}, m)$.

(a) *Define $h_{x,t}(y) := f(y)g_t(x-y)$. Prove that $h_{x,t} \in L^1$ for all $x \in \mathbb{R}$ and $t \neq 0$.*

(b) *Put*

$$f * g_t(x) := \int_{\mathbb{R}} h_{x,t}(y)\, dm(y).$$

Show that

$$\| f * g_t \|_1 \leq \| f \|_1$$

(c) *Suppose that $|f| \leq C < \infty$ and $f(x) = 0$ for $|x| > C$. Prove that*

$$\lim_{t \to 0} \| f * g_t - f \|_1 = 0 \tag{5}$$

(d) *Prove that (5) holds for all $f \in L^1$.*

Solution. a) As the product of measurable functions, $h_{x,t}$ is measurable. Since $|g_t| \leq \frac{3}{4t}$, it follows that $|h_{x,t}| \leq \frac{3}{4t}|f|$. Since f is integrable it follows that $h_{x,t}$ is too.

b) Consider the function

$$F_t(x, y) := h_{x,t}(y) = f(y)g_t(x-y) = G_t(x-y, y)$$

where

$$G_t(z, y) := f(y)g_t(z).$$

As the product of measurable functions on the coordinates, the latter is clearly measurable on $\mathbb{R}^2$, and hence so is F_t since it is obtained from G_t by precomposition with a linear transformation T (Folland, Theorem 2.44 a.). Therefore, by Fubini-Tonelli, $f * g_t = \int F_t(\cdot, y)\, dy$ is measurable for a.e. y.

Since $\det T = 1$, in fact

$$\begin{aligned}
\| f * g_t \|_1 = \int_{\mathbb{R}} |f * g_t(x)|\, dx &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y) g_t(x-y)\, dy \right| dx \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(y)g_t(x-y)|\, dy\, dx \\
&= \int_{\mathbb{R}^2} |F_t(x, y)|\, dx\, dy \\
&= \int |G_t(z, y)|\, dz\, dy \\
&= \int |f| \int |g_t| = \| f \|_1
\end{aligned}$$

in view of the direct calculation $\int |g_t| = \int g_t = 1$.

c) Since $|f| \leq C$, it follows that

$$|f * g_t(x)| = \left| \int f(y) g_t(x-y)\, dy \right| \leq \int |f(y)g_t(x-y)|\, dy \leq C \int |g_t(x-y)|\, dy = C.$$

Furthermore, if $|x| > C + |t|$ then $g_t(x-y) = 0$ for $|x-y| > |t|$. But if $|x-y| \leq |t|$ then $|y| \geq |x| - t > C$, so $f(y) = 0$. It follows that in this case $f(y)g_t(x-y) \equiv 0$ for all y, and therefore

$$f * g_t(x) = \int f(y)g_t(x-y)\,dy = 0.$$

Thus $|f * g_t| \leq C \cdot 1_{[-C-1,C+1]}$ for $|t| < 1$, and $|f * g_t - f| \leq 2C \cdot 1_{[-C-1,C+1]}$, and if we can show that $f * g_t \to f$ a.e. then the dominated convergence theorem applies to give the result.

Now suppose x is a Lebesgue point of f, and therefore a point of approximate continuity of f. Thus given $\epsilon > 0$ there is $r_0 > 0$ so that for $0 < r < r_0$

$$m(E_{r,\epsilon}) < r\epsilon.$$

where

$$E_{r,\epsilon} = \{y \in (x-r, x+r) : |f(y) - f(x)| > \epsilon\}$$

Then for $t < r_0$

$$\begin{aligned}
|f * g_t(x) - f(x)| &= \left| \int f(y)g_t(x-y)\,dy - f(x) \right| \\
&= \left| \int (f(y) - f(x))g_t(x-y)\,dy \right| \\
&\leq \int |f(y) - f(x)|g_t(x-y)\,dy \\
&= \int_{E_{t,\epsilon}} |f(y) - f(x)|g_t(x-y)\,dy + \int_{[x-t,x+t]\setminus E_{t,\epsilon}} |f(y) - f(x)|g_t(x-y)\,dy \\
&\leq \int_{E_{t,\epsilon}} (2C)\frac{3}{4t}\,dy + \int_{[x-t,x+t]\setminus E_{t,\epsilon}} \epsilon g_t(x-y)\,dy \\
&\leq \frac{3C}{2t}t\epsilon + \epsilon = \epsilon\left(\frac{3C}{2} + 1\right).
\end{aligned}$$

Therefore $f * g_t(x) \to f(x)$ as $t \to 0$ for a.e. x, and the dominated convergence theorem applies.

d) For $C < \infty$ put $f_C := f \cdot 1_{\{|f| \leq C\} \cap [-C,C]}$. Then dominated convergence implies that $\| f_C - f \|_1 \to 0$ as $C \to \infty$. Let $\epsilon > 0$ be given and let C be so large that $\| f_C - f \|_1 < \epsilon$. By c), we may take δ small enough that if $|t| < \delta$ then $\| f_C * g_t - f_C \|_1 < \epsilon$. Then for such t we have by parts b) and c)

$$\begin{aligned}
\| f * g_t - f \|_1 &\leq \| f * g_t - f_C * g_t \|_1 + \| f_C * g_t - f_C \|_1 + \| f_C - f \|_1 \\
&= \| (f - f_C) * g_t \|_1 + \| f_C * g_t - f_C \|_1 + \| f_C - f \|_1 \\
&\leq \| (f - f_C) \|_1 + \epsilon + \epsilon \\
&< 3\epsilon.
\end{aligned}$$