

MATH 8100 COURSE NOTES II: SOME EXOTIC SETS

The classical Cantor set. The prototype of this class of sets is the classical Cantor set, obtained by the following iterative process. Let $I = [0, 1]$, and let $D_1 := I \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Repeat this construction with each of these two constituent intervals, removing the open interval of $\frac{1}{3}$ its length, and so on. This gives us a decreasing nested sequence of closed sets $D_1 \supset D_2 \supset \dots$. The classical Cantor set is the residual closed set $C := \bigcap_{i=1}^{\infty} D_i$. To see that $C \neq \emptyset$ it is enough to note that any nested sequence of closed subsets of a compact metric space (in this case $[0, 1]$) is nonempty.

One nice way to describe C is as follows. Note that any $x \in [0, 1]$ can be expressed as

$$x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{3^i} \tag{1}$$

where each $\epsilon_i \in \{0, 1, 2\}$ —the *ternary expansion* of x . The ternary expansion of x is unique unless x is a rational number with denominator equal to a power of 3, in which case it admits exactly two such expansions: one with all $\epsilon_i = 0$ from some point on, and one with all $\epsilon_i = 2$ from some point on. It is easy to see that D_1 consists of all $x \in [0, 1]$ that admit a ternary expansion with $\epsilon_1 \neq 1$. Similarly D_2 consists of all x admitting a ternary expansion with $\epsilon_1, \epsilon_2 \neq 1$, and so on for D_3, D_4 , etc. Thus C consists of all x that admit a ternary expansion with all $\epsilon_i \neq 1$.

(By the way, this last characterization permits us to construct the *Cantor function*: if $C \ni x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{3^i}$ as above, with all $\epsilon_i = 0$ or 1, put $f(x) := \sum_{i=1}^{\infty} \frac{\epsilon_i}{2^{i+1}}$. This is clearly a monotonically increasing function on C , with $f(0) = 0$ and $f(1) = 1$. For general $x \in [0, 1]$, put $f(x) := \sup_{y \leq x, y \in C} f(y)$.)

The Cantor set has measure zero: for $[0, 1] \setminus C$ is the union of a countable disjoint family of open intervals $(\frac{1}{3}, \frac{2}{3}) \cup (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}) \cup \dots$, i.e. 2^{i-1} intervals of length $(\frac{1}{3})^i, i = 1, 2, \dots$. Since C is closed, it is measurable, so by countable additivity,

$$1 = m([0, 1]) = m(C) + m([0, 1] \setminus C) = m(C) + \sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i} = m(C) + 1.$$

Thus $m(C) = 0$.

Cantor-type sets of positive measure. Fix $0 < \alpha \leq \frac{1}{3}$. We repeat the process above, but with a slight difference: at the i th stage we remove the middle open interval of length α^i from each remaining interval. Thus we put $E_1 := [0, 1] \setminus (\frac{1-\alpha}{2}, \frac{1+\alpha}{2}) = [0, \frac{1-\alpha}{2}] \cup [\frac{1+\alpha}{2}, 1]$, $E_2 := [0, \frac{1-\alpha-2\alpha^2}{4}] \cup$

$[\frac{1-\alpha+2\alpha^2}{4}, \frac{1-\alpha}{2}] \cup [\frac{1+\alpha}{2}, 1-\alpha^2] \cup [\frac{1+\alpha+2\alpha^2}{2}, 1]$. It is easy to check that each of the 2^i intervals comprising E_i has length $\frac{1-\sum_{j=1}^i 2^{i-1}\alpha^j}{2^i} = \frac{1-3\alpha+2^i\alpha^{i+1}}{2^i(1-2\alpha)}$. Put $C_\alpha := \bigcap_{i=1}^\infty E_i$.

Exercise 1. What are the endpoints of the intervals comprising E_i ?

Thus by the downward continuity of m

$$m(E_i) = \frac{1-3\alpha+2^i\alpha^{i+1}}{1-2\alpha} \rightarrow \frac{1-3\alpha}{1-2\alpha} = m(C_\alpha)$$

Exercise 2. What goes wrong if $\alpha > \frac{1}{3}$?

All of the sets C_α have the following properties:

- (1) C_α is totally disconnected, i.e. given any two distinct points $x, y \in C_\alpha$ there is $z \notin C_\alpha$ lying between them.
- (2) Every point of C_α is a limit point of C_α .

These two properties imply

Fact 1. Given any point $x \in C_\alpha$, and any neighborhood U of x in \mathbb{R} , there is a maximal interval $I \subset [0, 1] \setminus C_\alpha$ such that $I \subset U$.

Taking it to the limit. Consider a sequence $(\alpha_1, \alpha_2, \dots) \in (0, \frac{1}{3})$ and put $\beta_i := \frac{1-3\alpha_i}{1-2\alpha_i}$. We suppose that $\sum_i \beta_i$ converges (in particular, $\alpha_i \rightarrow \frac{1}{3}$).

We construct a new set \mathfrak{C}_α as follows. Put $\mathfrak{C}_1 := C_{\alpha_1}$. Thus $[0, 1] \setminus \mathfrak{C}_1$ is a countable union of open intervals of total length $1 - \beta_1$. We insert a scaled-down copy of C_{α_2} into the closure of each of these intervals. These new sets overlap \mathfrak{C}_1 in a countable set, and together they occupy the proportion β_2 of the measure of the complement of \mathfrak{C}_1 , i.e. their total measure is $(1 - \beta_1)\beta_2$. Put \mathfrak{C}_2 to be the union of \mathfrak{C}_1 and these countably many scaled copies of C_{α_2} . Then $m(\mathfrak{C}_2) = \beta_1 + (1 - \beta_1)\beta_2$, and

$$m([0, 1] \setminus \mathfrak{C}_2) = 1 - \beta_1 - \beta_1(1 - \beta_2) = (1 - \beta_1)(1 - \beta_2).$$

Again the complement of \mathfrak{C}_2 is a countable union of open intervals, and into the closure of each of these we insert a scaled-down copy of C_{α_3} to obtain a new set \mathfrak{C}_3 such that

$$\begin{aligned} m([0, 1] \setminus \mathfrak{C}_3) &= 1 - (m(\mathfrak{C}_2) + (1 - m(\mathfrak{C}_2))\beta_3) \\ &= 1 - (\beta_1 + (1 - \beta_1)\beta_2 + (1 - \beta_1)(1 - \beta_2)\beta_3) \\ &= (1 - \beta_1)(1 - \beta_2)(1 - \beta_3). \end{aligned}$$

Continuing in this way we construct $\mathfrak{C}_1 \subset \mathfrak{C}_2 \subset \dots$ such that

$$1 - m(\mathfrak{C}_k) = \prod_{i=1}^k (1 - \beta_i).$$

Put $\mathfrak{C} := \bigcup \mathfrak{C}_k$. Then by continuity

$$0 < m(\mathfrak{C}) = 1 - \prod_{i=1}^\infty (1 - \beta_i) < 1$$

in view of the convergence of the series $\sum \beta_i$. By the same token, if J is any one of the intervals comprising $[0, 1] \setminus \mathfrak{C}_k$ then

$$0 < m(J \cap \mathfrak{C}) = m(J) \left(1 - \prod_{i=k+1}^{\infty} (1 - \beta_i)\right) < m(J). \quad (2)$$

It is clear (!) that \mathfrak{C} is dense in $[0, 1]$.

We claim that for any nondegenerate interval $I \subset [0, 1]$, both $m(I \cap \mathfrak{C})$, $m(I \setminus \mathfrak{C}) > 0$. In view of (2) it is enough to show that any such I includes one of the maximal complementary intervals of some \mathfrak{C}_k . Since \mathfrak{C} is dense in $[0, 1]$, there is some $x \in \mathfrak{C}$ lying in the interior U of I . Let k be the smallest value such that $x \in \mathfrak{C}_k$. Thus x lies in one of the countably many scaled copies of C_{α_k} inserted in the complementary intervals of \mathfrak{C}_{k-1} . By Fact 1, one of the maximal complementary intervals of this scaled copy is a subset of U . This establishes the claim.