MATH 8100 COURSE NOTES III: A NONMEASURABLE SET

The following is the standard example of a set $S \subset \mathbb{R}$ that is not measurable with respect to Lebesgue measure.

For $x, y \in \mathbb{R}$, say that $x \sim y$ if $x - y \in \mathbb{Q}$. Clearly $\sim$ is an equivalence relation. Let $S \subset \mathbb{R}$ be a set that contains exactly one element from every $\sim$ equivalence class (the axiom of choice dictates that such a set exists). Thus

(1) For every $x \in \mathbb{R}$ there is exactly one $s \in S$ such that $x - s \in \mathbb{Q}$.
(2) If $s, t \in S$ and $s \neq t$ then $s - t \notin \mathbb{Q}$.

Suppose that S is measurable. We will derive a contradiction.

First, by property (1) above, every $x \in \mathbb{R}$ belongs to some set $S + q$ for some $q \in \mathbb{Q}$, i.e.

$$\mathbb{R} = \bigcup_{q \in \mathbb{Q}} (S + q).$$

Thus by countable subadditivity $\infty = m(\mathbb{R}) \leq \sum_{q \in \mathbb{Q}} m(S+q)$. Since $m(S+q) = m(S)$ for all $q \in \mathbb{Q}$, this implies that $m(S) > 0$.

Let $r \in \mathbb{Q} \cap (0, m(S))$, i.e r is a positive rational number and

$$m(S) > r.$$

For each $n \in \mathbb{Z}$ put $T'_n := S \cap [nr, (n+1)r)$. Then S is the disjoint union of the T'_n, and all of these sets are measurable, so by countable additivity $m(S) = \sum_{n \in \mathbb{Z}} m(T'_n)$. Furthermore $T_n := T'_n - nr \subset [0, r)$, with

$$m(T_n) = m(T'_n).$$

Finally, we observe that the T_n are pairwise disjoint, for if $x \in T_l \cap T_n$ then

$$x = s - lr = t - nr$$

for some $s, t \in S$ (in fact $s \in T'_l, t \in T'_n$). Thus $s - t = lr - nr \in \mathbb{Q}$, so by property (2) above $s = t$ and therefore $l = n$.

Thus by monotonicity and countable additivity

$$m(S) > r = m([0, r)) \geq m\left(\bigcup_n T_n\right) = \sum_n m(T_n) = \sum_n m(T'_n) = m(S),$$

which is a contradiction.