MATH 8100 COURSE NOTES I: THE RIEMANN INTEGRAL

J.H.G. FU

Introduction. The Riemann integral is conceptually the simplest approach to the integral. Nevertheless a closer look at it leads to some ideas that indicate a more penetrating view. These ideas will occupy us for most of the semester.

0.1. **Definition of the Riemann integral.** A **partition** of a closed bounded interval $[a, b]$ is a finite subset of $[a, b]$ that contains both endpoints. We will always index the elements of a partition $\mathcal{P}$ in increasing order, writing e.g.

$$\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_N = b\}.$$

If $\mathcal{Q} \supset \mathcal{P}$ is another partition of $[a, b]$ then we say that $\mathcal{Q}$ is a **refinement** of $\mathcal{P}$.

Let f be a bounded function on $[a, b]$. Given a partition $\mathcal{P}$ as above, we put

$$U(f, \mathcal{P}) := \sum_{i=1}^{N} (x_i - x_{i-1}) \sup_{[x_{i-1}, x_i]} f,$$

$$L(f, \mathcal{P}) := \sum_{i=1}^{N} (x_i - x_{i-1}) \inf_{[x_{i-1}, x_i]} f,$$

the **upper** and **lower Riemann sums** for $f, \mathcal{P}$. Clearly

$$U(f, \mathcal{P}) \geq L(f, \mathcal{P}). \tag{1}$$

Lemma 1. (1) *If $\mathcal{Q}$ is a refinement of $\mathcal{P}$ then*

$$U(f, \mathcal{Q}) \leq U(f, \mathcal{P}), \quad L(f, \mathcal{Q}) \geq L(f, \mathcal{P})$$

(2) *Given any two partitions $\mathcal{P}, \mathcal{Q}$ of $[a, b]$,*

$$L(f, \mathcal{P}) \leq U(f, \mathcal{Q}).$$

Proof. To prove part (1) we may assume that $\mathcal{Q}$ contains exactly one more point than $\mathcal{P}$, in which case it is obvious.

To prove (2), let $\mathcal{R} := \mathcal{P} \cup \mathcal{Q}$. Then $\mathcal{R}$ refines both $\mathcal{P}$ and $\mathcal{Q}$, so by part (1) and equation (1)

$$U(f, \mathcal{P}) \geq U(f, \mathcal{R}) \geq L(f, \mathcal{R}) \geq L(f, \mathcal{Q}).$$

□

Put $\overline{\int}_a^b f(x)\,dx$ for the infimum of the set of all $U(f,\mathcal{P})$ as $\mathcal{P}$ ranges over all partitions of $[a,b]$, and similarly $\underline{\int}_a^b f(x)\,dx$ for the supremum of the $L(f,\mathcal{P})$. By part (2) of Lemma 1,

$$\overline{\int_a^b} f(x)\,dx \geq \underline{\int_a^b} f(x)\,dx.$$

If these two quantities agree then we put $\int_a^b f(x)\,dx$ for their common value, and say that f is **Riemann integrable**.

Proposition 2. *f is Riemann integrable iff for every $\epsilon > 0$ there exists a partition $\mathcal{P}$ of $[a,b]$ such that*

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon.$$

Proof. Suppose f is Riemann integrable. Then there are partitions $\mathcal{Q},\mathcal{R}$ such that

$$\int f - \frac{\epsilon}{2} < L(f,\mathcal{Q}) \leq U(f,\mathcal{R}) < \int f + \frac{\epsilon}{2}.$$

Taking $\mathcal{P} := \mathcal{Q} \cup \mathcal{R}$,

$$\int f - \frac{\epsilon}{2} < L(f,\mathcal{Q}) \leq L(f,\mathcal{P}) \leq U(f,\mathcal{P}) \leq U(f,\mathcal{R}) < \int f + \frac{\epsilon}{2}.$$

The converse is obvious. □

0.2. **Continuity, discontinuity and oscillation.** For $x \in [a,b]$, put

$$\operatorname{osc}(x) := \lim_{\epsilon \downarrow 0} \left[\sup_{(x-\epsilon,x+\epsilon)} f - \inf_{(x-\epsilon,x+\epsilon)} f \right].$$

The limit exists since the quantity in brackets decreases monotonically as $\epsilon \downarrow 0$. Observe that f is continuous at x iff $\operatorname{osc}(x) = 0$.

Lemma 3. *If $c < x < d$ then*

$$\sup_{(c,d)} f - \inf_{(c,d)} f \geq \operatorname{osc}(x).$$

Lemma 4. osc *is upper semicontinuous, i.e. if $x_i \to x$ then*

$$\operatorname{osc}(x) \geq \limsup_{i\to\infty} \operatorname{osc}(x_i).$$

Equivalently, $\operatorname{osc}^{-1}[c,\infty)$ *is a closed set for every $c \geq 0$.*

0.3. **Measure zero and outer measure.** We say that a subset $S \subset [a,b]$ **has measure zero** if, given $\epsilon > 0$, there are $\alpha_i < \beta_i, i = 1,2,\ldots$ such that

$$S \subset \bigcup_{i=1}^{\infty} (\alpha_i,\beta_i)$$

and

$$\sum_{i=1}^{\infty} (\beta_i - \alpha_i) < \epsilon.$$

Lemma 5. *If S has measure zero and $S' \subset S$, then S' has measure zero too.* $\square$

Lemma 6. *Suppose $S = \bigcup_{i=1}^{\infty} S_i$. If each S_i has measure zero then so does S.*

Proof. Given $\epsilon > 0$, take $\alpha_{ij} < \beta_{ij}$ so that

$$S_i \subset \bigcup_{j=1}^{\infty} (\alpha_{ij}, \beta_{ij}), \quad \sum_{j=1}^{\infty} (\beta_{ij} - \alpha_{ij}) < 2^{-i}\epsilon, \quad i = 1, 2, \ldots.$$

Then $S \subset \bigcup_{i,j} (\alpha_{ij}, \beta_{ij})$, with

$$\sum_{i,j} (\beta_{ij} - \alpha_{ij}) = \sum_i \sum_j (\beta_{ij} - \alpha_{ij}) < \sum_i 2^{-i}\epsilon = \epsilon.$$

$\square$

More generally, the **outer measure** of a set $S \subset \mathbb{R}$ is defined to be

$$m^*(S) := \inf \left\{ \sum_{i=1}^{\infty} (\beta_i - \alpha_i) : S \subset \bigcup_{i=1}^{\infty} (\alpha_i, \beta_i) \right\}.$$

Thus S has measure zero iff $m^*(S) = 0$.

Exercise.

$$m^*(S) = \inf \left\{ \sum_{i=1}^{\infty} (\beta_i - \alpha_i) : S \subset \bigcup_{i=1}^{\infty} [\alpha_i, \beta_i] \right\}.$$

Proposition 7. *If S' has measure zero then $m^*(S \cup S') = m^*(S)$.*

0.4. Characterization of Riemann integrable functions.

Theorem 8. *The bounded function f is Riemann integrable iff the set of points of discontinuity of f has measure zero.*

Proof. Put D for the set of discontinuities of f, and $D_n := \{x : \operatorname{osc}(x) \geq \frac{1}{n}\}$. Thus $D = \bigcup_{n=1}^{\infty} D_n$. By Lemmas 5 and 6, D has measure zero iff every D_n does.

Suppose first that f is Riemann integrable. Let $\epsilon > 0$ and $n \in \mathbb{N}$ be given. By Prop. 2, there exists a partition $\mathcal{P}$ of $[a, b]$ such that $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \frac{\epsilon}{n}$. Let $[\alpha_1, \beta_1], \ldots, [\alpha_N, \beta_N]$ be the intervals of $\mathcal{P}$ whose interiors meet D_n. Then

$$\begin{aligned} \frac{\epsilon}{n} > U(f, \mathcal{P}) - L(f, \mathcal{P}) &\geq \sum_{j=1}^{N} \Big(\sup_{(\alpha_j, \beta_j)} f - \inf_{(\alpha_j, \beta_j)} f \Big)(\beta_j - \alpha_i) \\ &\geq \frac{1}{n} \sum_{j=1}^{N} (\beta_j - \alpha_j) \end{aligned}$$

It follows from Prop. 7 that $m^*(D_n) < \epsilon$. Since ϵ was arbitrarily chosen, we conclude that D_n has measure zero. Since this is true for every $n \in \mathbb{N}$, the same is true of D.

Suppose next that D has measure zero. Let $\epsilon > 0$ be given, and let $n_0 > \frac{2(b-a)}{\epsilon}$. By Lemma 4, the set D_{n_0} is a compact set of measure zero. Hence there is a *finite* cover $(\alpha_1, \beta_1), \ldots, (\alpha_N, \beta_N)$ of D_{n_0} such that $\sum_{i=1}^N (\beta_i - \alpha_i) < \frac{\epsilon}{4C}$, where $C := \sup |f|$. On the other hand, every point $x \in [a, b]$ that does not lie in one of these intervals has $\operatorname{osc}(x) < \frac{1}{n_0}$, and hence admits a neighborhood (γ_x, δ_x) such that $\sup_{(\gamma_x,\delta_x)} f - \inf_{(\gamma_x,\delta_x)} f < \frac{1}{n_0} < \frac{\epsilon}{2(b-a)}$. Since the set of such x is compact, we may cover it by finitely many such intervals.

Let $\mathcal{P}$ be the partition of $[a, b]$ consisting of all $\alpha, \beta, \gamma, \delta$ above that lie in $[a, b]$. Then every interval of $\mathcal{P}$ is included either in an interval (γ, δ) or an interval (α, β) (or both). Let $\mathcal{I}$ denote the family of intervals of $\mathcal{P}$ of the latter type, and $\mathcal{J}$ the rest. Thus every interval of $\mathcal{J}$ is included in an interval (γ, δ). Then, denoting by $|I| := d - c$ the length of an interval $I = [c, d]$,

$$
\begin{aligned}
U(f, \mathcal{P}) - L(f, \mathcal{P}) &= \sum_{I \in \mathcal{I}} (\sup_I f - \inf_I f)|I| + \sum_{J \in \mathcal{J}} (\sup_J f - \inf_J f)|J| \\
&\le 2C \sum_{I \in \mathcal{I}} |I| + \frac{\epsilon}{2(b-a)} \sum_{J \in \mathcal{J}} |J| \\
&< 2C \frac{\epsilon}{4C} + \frac{\epsilon}{2(b-a)}(b-a) = \epsilon.
\end{aligned}
$$

Thus f is Riemann integrable, by Prop. 2. □

A careful look at the proof above reveals

Corollary 9. *If* $m^*(\{x : \operatorname{osc}(x) \ge B\}) \ge A$, *then* $\overline{\int} f - \underline{\int} f \ge AB$.