

AMPLENESS OF INTERSECTIONS OF TRANSLATES OF THETA DIVISORS IN AN ABELIAN FOURFOLD

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INTRODUCTION

Varieties with ample cotangent bundle are interesting from many points of view. If X is such a variety, defined over a field \mathbf{k} ,

- (geometric) all subvarieties of X are of general type and there are only finitely many rational maps from any fixed projective variety to X ([NS]);
- (analytic) if $\mathbf{k} = \mathbf{C}$, any holomorphic map $\mathbf{C} \rightarrow X$ is constant ([De], (3.1.));
- (arithmetic) if \mathbf{k} is a number field, the set of \mathbf{k} -rational points of X is conjectured to be finite (see [Mo]; the analogous statement over function fields of curves is known to hold by [N] or [MD]).

However, there are relatively few concrete examples of such varieties. Bogomolov was the first to give a general procedure to produce such examples (his construction is explained in [D]). In that article, more examples are constructed: it is shown that given a principally polarized abelian variety (A, Θ) , an integer $n \geq \frac{1}{2} \dim A$, and sufficiently ample (i.e., algebraically equivalent to sufficiently high multiples of Θ) general divisors D_1, \dots, D_n , the smooth variety $D_1 \cap \dots \cap D_n$ has ample cotangent bundle. In this paper we prove an analogous result for general abelian fourfolds. We work over an algebraically closed field \mathbf{k} .

Theorem 1. *Let (A, Θ) be a general principally polarized abelian fourfold. For $a \in A$ general, the smooth surface $\Theta \cap \Theta_a$ has ample cotangent bundle.*

Here Θ_a denotes the translate $\Theta + a$ of Θ by a . Our proof shows that the conclusion of the theorem already holds on a general Jacobian fourfold.

1. THE AMPLENESS OF $\Omega_{\Theta \cap \Theta_a}$

Let (A, Θ) be a principally polarized abelian fourfold. Assume $\Theta \cap \Theta_a$ is a smooth surface. The cotangent bundle $\Omega_{\Theta \cap \Theta_a}$ fits into the exact sequence

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of conormal and cotangent bundles

$$0 \longrightarrow \mathcal{O}_{\Theta \cap \Theta_a}(-\Theta) \oplus \mathcal{O}_{\Theta \cap \Theta_a}(-\Theta_a) \longrightarrow \Omega_A|_{\Theta \cap \Theta_a} \longrightarrow \Omega_{\Theta \cap \Theta_a} \longrightarrow 0.$$

Being a quotient of a trivial vector bundle, it is generated by its global sections, which are identified with $H^0(\Theta \cap \Theta_a, \Omega_A|_{\Theta \cap \Theta_a}) \simeq H^0(A, \Omega_A)$. To show that $\Omega_{\Theta \cap \Theta_a}$ is ample, we must show that the associated morphism

$$(1) \quad \psi_a : \mathbf{P}(\Omega_{\Theta \cap \Theta_a}) \longrightarrow \mathbf{P}(H^0(A, \Omega_A))$$

is finite (we follow Grothendieck's notation: given a coherent sheaf \mathcal{E} on a scheme, $\mathbf{P}(\mathcal{E}) = \text{Proj}(\text{Sym } \mathcal{E})$). More concretely, a point in $\mathbf{P}(H^0(A, \Omega_A))$ corresponds to a hyperplane in $H^0(A, \Omega_A)$, or to a line ℓ in $T_{A,0}$, and

$$\mathbf{P}(\Omega_{\Theta \cap \Theta_a}) = \{(\ell, x) \in \mathbf{P}(H^0(A, \Omega_A)) \times (\Theta \cap \Theta_a) \mid \ell \subset T_{\Theta \cap \Theta_a, x}\}$$

where $T_{A,0}$ and $T_{A,x}$ are identified by translation by x , and ψ_a is the first projection.

In other words, to prove that $\Omega_{\Theta \cap \Theta_a}$ is ample, we must prove that, for any nonzero $\partial' \in T_{A,0}$, the set of points $x \in \Theta \cap \Theta_a$ such that $\partial' \in T_{\Theta \cap \Theta_a, x}$ is finite. We denote by $[\partial']$ the point of $\mathbf{P}(H^0(A, \Omega_A))$ determined by ∂' .

2. THE DIVISOR $\Theta \cap \partial\Theta$

Let (A, Θ) be a principally polarized abelian variety and let θ be a nonzero section of $\mathcal{O}_A(\Theta)$. We define, for any ∂ in $T_{A,0}$, a section $\partial\theta$ of $\mathcal{O}_\Theta(\Theta)$ by the requirement that for any open set U of A and any trivialization $\varphi : \mathcal{O}_U \xrightarrow{\sim} \mathcal{O}_\Theta(\Theta)|_U$, we have $\partial\theta = \varphi(\partial(\varphi^{-1}(\theta)))|_\Theta$ in $U \cap \Theta$. We denote its zero locus by $\Theta \cap \partial\Theta$. Set-theoretically, $\Theta \cap \partial\Theta$ is the set of points x of Θ where the Zariski tangent space $T_{\Theta, x}$ contains ∂ .

The differential of the isomorphism $A \rightarrow \text{Pic}^0(A)$ induced by the polarization Θ identifies $T_{A,0}$ with $T_{\text{Pic}^0(A),0} \simeq H^1(A, \mathcal{O}_A)$. The exact sequence

$$0 \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{O}_A(\Theta) \longrightarrow \mathcal{O}_\Theta(\Theta) \longrightarrow 0$$

yields a composed isomorphism

$$(2) \quad H^0(\Theta, \mathcal{O}_\Theta(\Theta)) \xrightarrow{\sim} H^1(A, \mathcal{O}_A) \xrightarrow{\sim} T_{A,0}$$

whose inverse is given by

$$\partial \longmapsto \partial\theta$$

When A has dimension 4, the ampleness of the cotangent bundle of $\Theta \cap \Theta_a$ is equivalent to the following: *for all nonzero $\partial' \in T_{A,0}$, the scheme $\Theta \cap \partial'\Theta \cap \Theta_a \cap \partial'\Theta_a$ is finite.*

For $\partial \neq 0$, the scheme $\Theta \cap \partial\Theta$ is a limit of intersections of translates of Θ in the following sense. Let $m : \Theta \times A \rightarrow A$ be the morphism $(x, y) \mapsto x - y$ and let $\mathcal{T} = m^{-1}(\Theta)$. The first projection $\Theta \times A \rightarrow \Theta$ identifies $\Theta \cap \Theta_a$ with the fiber at a of the second projection

$$p : \mathcal{T} \longrightarrow A$$

If $\tilde{A} \rightarrow A$ is the blow-up of 0, with exceptional divisor $\mathbf{P}(\Omega_{A,0})$, and $\tilde{\mathcal{F}}$ is the strict transform of \mathcal{F} in $\Theta \times \tilde{A} \rightarrow \Theta \times A$, we obtain a family

$$\tilde{p} : \tilde{\mathcal{F}} \longrightarrow \tilde{A}$$

whose fiber at $[\partial] \in \mathbf{P}(\Omega_{A,0})$ is isomorphic to $\Theta \cap \partial\Theta$. If Θ is irreducible, this is a flat family of subschemes of codimension 2 of A .

We will study the ampleness of the cotangent bundle of $\Theta \cap \Theta_a$ by letting it specialize to $\Theta \cap \partial\Theta$. More precisely, if we consider

$$\mathbf{P} = \{(\ell, x, \tilde{a}) \in \mathbf{P}(H^0(A, \Omega_A)) \times \tilde{\mathcal{F}} \mid \ell \subset T_{\tilde{p}^{-1}(\tilde{a}), (x, \tilde{a})}\}$$

the projection $\psi : \mathbf{P} \rightarrow \mathbf{P}(H^0(A, \Omega_A)) \times \tilde{A}$ restricts to ψ_a over $\mathbf{P}(H^0(A, \Omega_A)) \times \{a\}$, for $a \in A$ nonzero, and to a morphism

$$\psi_\partial : \mathbf{P}_\partial = \{(\ell, x) \in \mathbf{P}(H^0(A, \Omega_A)) \times (\Theta \cap \partial\Theta) \mid \ell \subset T_{\Theta \cap \partial\Theta, x}\} \rightarrow \mathbf{P}(H^0(A, \Omega_A))$$

over $\mathbf{P}(H^0(A, \Omega_A)) \times \{[\partial]\}$, for $\partial \in T_{A,0}$ nonzero. If ψ_∂ is *finite*, the same will be true for ψ_a for general a in A .

3. THE FINITENESS OF $\Theta \cap \partial\Theta \cap \partial'\Theta \cap \partial\partial'\Theta$

Let again (A, Θ) be a principally polarized abelian fourfold. As explained above, we would like to find a nonzero element ∂ of $T_{A,0}$ such that the morphism

$$\psi_\partial : \mathbf{P}_\partial \longrightarrow \mathbf{P}(H^0(A, \Omega_A))$$

is finite. If ∂' is a nonzero element of $T_{A,0}$, we may define as above a section $\partial\partial'\theta$ of $\mathcal{O}_{\Theta \cap \partial\Theta \cap \partial'\Theta}(\Theta)$ whose zero locus we denote by $\Theta \cap \partial\Theta \cap \partial'\Theta \cap \partial\partial'\Theta$ and which is isomorphic to the fiber $\psi_\partial^{-1}([\partial'])$.

Unfortunately, this scheme has codimension at most 3 for $\partial' = \partial$. We will first prove that for A a general Jacobian and ∂ general in $T_{A,0}$, this is the only positive-dimensional fiber of ψ_∂ .

Let C be a smooth curve of genus 4, take $A = \text{Pic}^3 C$, and let $\Theta \subset A$ be Riemann's theta divisor parametrizing effective divisors of degree 3 on C .

Proposition 2. *For C and ∂ general, all fibers of the morphism ψ_∂ are finite, except for that of $[\partial]$.*

Proof. Take $\partial \in T_{A,0}$ nonzero and let S_∂ be the (local complete intersection) surface $\Theta \cap \partial\Theta$. Noting that the restriction $H^1(A, \mathcal{O}_A) \rightarrow H^1(\Theta, \mathcal{O}_\Theta)$ is bijective and using the isomorphism (2), we obtain from the exact sequence

$$0 \longrightarrow \mathcal{O}_\Theta \xrightarrow{\partial\theta} \mathcal{O}_\Theta(\Theta) \longrightarrow \mathcal{O}_{S_\partial}(\Theta) \longrightarrow 0$$

an exact sequence

$$0 \longrightarrow \mathbf{k} \xrightarrow{\partial} T_{A,0} \longrightarrow H^0(S_\partial, \mathcal{O}_{S_\partial}(\Theta))$$

$$\partial' \longmapsto \partial'\theta.$$

Let $\partial' \in T_{A,0}$ be nonzero. If $\Gamma = \Theta \cap \partial\Theta \cap \partial'\Theta$ is an integral, i.e., irreducible and reduced, curve, $\partial'\theta$ is not a zero divisor in \mathcal{O}_{S_∂} and again, since $H^1(A, \mathcal{O}_A) \rightarrow H^1(S_\partial, \mathcal{O}_{S_\partial})$ is bijective, we obtain from the exact sequence

$$0 \longrightarrow \mathcal{O}_{S_\partial} \xrightarrow{\partial'\theta} \mathcal{O}_{S_\partial}(\Theta) \longrightarrow \mathcal{O}_\Gamma(\Theta) \longrightarrow 0$$

a coboundary map $H^0(\Gamma, \mathcal{O}_\Gamma(\Theta)) \rightarrow T_{A,0}$ that sends $\partial\partial'\theta$ to ∂ . This section is in particular nonzero hence its zero locus $\Theta \cap \partial\Theta \cap \partial'\Theta \cap \partial\partial'\Theta$ is finite, which is what we need to prove.

We assume from now on that C is not hyperelliptic and we identify it with its canonical model in $\mathbf{P}^3 = \mathbf{P}(H^0(C, \omega_C)) = \mathbf{P}(H^0(A, \Omega_A))$, where it is the intersection of a quadric Q (which will be assumed to be smooth) and a cubic.

The projectivization of the tangent space to Θ at a point corresponding to a divisor D of degree 3 such that $h^0(C, D) = 1$ is the plane spanned in \mathbf{P}^3 by the points of D . The underlying reduced curve Γ_{red} therefore parametrizes effective divisors of degree 3 on C that lie in a plane that contains the line $\ell_{\partial, \partial'}$ spanned by $[\partial]$ and $[\partial']$. We will distinguish several cases depending on the relative positions of $[\partial]$, $[\partial']$, and C in \mathbf{P}^3 .

We first introduce some notation, following [I]: given a pencil g_e^1 on C with reduced base locus, we define, for any $d \in \{1, \dots, e\}$, a reduced curve in the d -th symmetric power $C^{(d)}$ by setting

$$X_d(g_e^1) = \{p_1 + \dots + p_d \in C^{(d)} \mid \exists D \in C^{(e-d)} \ D + p_1 + \dots + p_d \in g_e^1\}$$

3.1. Case 1: $\ell_{\partial, \partial'} \cap C = \emptyset$. The planes containing $\ell_{\partial, \partial'}$ cut on C the divisors of a base-point-free g_6^1 contained in $|\omega_C|$, and the curve Γ_{red} is the image in Θ of the curve $X_3(g_6^1) \subset C^{(3)}$. It follows from [ACGH], Lemma VIII.(3.2) that the cohomology class of Γ_{red} is $[\Theta]^3$, so Γ is reduced.

The associated map $\phi : C \rightarrow (g_6^1)^* = \mathbf{P}^1$ coincides with the projection of $C \subset \mathbf{P}^3$ from the line $\ell_{\partial, \partial'}$. The lemma below shows that the monodromy group G of ϕ is the full symmetric group \mathfrak{S}_6 . It implies that Γ is integral, and we are done in this case.

Lemma 3. *For C general and $[\partial] \notin Q$, the group G is 2-transitive and contains a simple transposition.*

Proof. The 2-transitivity of G is equivalent to the irreducibility of the curve $X_2(g_6^1)$ in $C^{(2)}$.

Let $\pi : C^2 \rightarrow C^{(2)}$ be the quotient map. For any divisor (resp. divisor class) D on C , let C_D be the unique divisor (resp. divisor class) on $C^{(2)}$ such that $\pi^*C_D = p_1^*D + p_2^*D$. Let δ be the unique divisor class on $C^{(2)}$ such that $\pi^*\delta$ is linearly equivalent to the diagonal of C^2 . We have the linear equivalence $X_2(g_6^1) \equiv C_{g_6^1} - \delta$ ([I], Lemma 2.1). Moreover, $\delta^2 = -3$ and $\delta \cdot C_D = \deg(D)$.

Assume C is sufficiently general so that the map

$$\begin{aligned} \text{Pic}(C) \oplus \mathbf{Z} &\longrightarrow \text{Pic}(C^{(2)}) \\ (D, b) &\longmapsto C_D - b\delta \end{aligned}$$

is bijective. If $X_2(g_6^1)$ is reducible, write the divisor class of a nontrivial union of components, say Y , as $C_D - b\delta$, so that the class of $X_2(g_6^1) - Y$ is $C_{g_6^1-D} - (1-b)\delta$. Replacing Y with $X_2(g_6^1) - Y$ if necessary, we may assume $b \geq 0$.

We now use [I], Appendix 6.1: for any divisor E on C , we have

$$H^0(C^{(2)}, C_E) \simeq \text{Sym}^2 H^0(C, E) \quad \text{and} \quad H^0(C^{(2)}, C_E - \delta) \simeq \bigwedge^2 H^0(C, E)$$

It follows that if E is effective and $h^0(C, E) = 1$, the linear system $|C_E - \delta|$ is empty, and $|C_E| = \{C_E\}$. Since our g_6^1 has no base points, $X_2(g_6^1)$ contains no such curve. It follows that D moves in a pencil, hence $\deg(D) \geq 3$ since C is not hyperelliptic. Since the diagonal is not a component of $X_2(g_6^1)$, we must have $(C_{g_6^1-D} - (1-b)\delta) \cdot \delta \geq 0$, hence $3b \leq 9 - \deg(D)$.

If $\deg(D) \geq 4$, we get $b \leq 1$ but this is absurd since $|C_{g_6^1-D} - (1-b)\delta|$ is then empty. Hence D is a g_3^1 and $b \leq 2$. By [I], Appendix 6.3, the vector subspace $H^0(C^{(2)}, C_{g_3^1} - 2\delta) \subset H^0(C^{(2)}, C_{g_3^1})$ is isomorphic to the space of quadratic forms that vanish on the image of $C \rightarrow (g_3^1)^*$, hence vanishes. We get $b \in \{0, 1\}$ and, replacing Y with $X_2(g_6^1) - Y$ if necessary, $Y \equiv C_{g_3^1} - \delta$. More precisely, $Y = X_2(g_3^1)$. The g_3^1 is given by one of the rulings of the quadric Q , hence $X_2(g_3^1)$ may be contained in $X_2(g_6^1)$ only if the line $\ell_{\partial, \partial'}$ meets all lines of this ruling. Since $[\partial]$ is not in Q , this cannot happen and $X_2(g_6^1)$ is irreducible.

To prove that G contains a simple transposition, we check that for C general, there is a point $p \in C$ such that $\phi : C \rightarrow \mathbf{P}^1$ ramifies simply at p and p is the only ramification point of ϕ in its fiber.

The degree of the ramification locus is $6 + 6 \cdot 2 = 18$. If all the ramification points are either nonsimple or their fiber contains other ramification points, the support of the branch locus of ϕ in \mathbf{P}^1 contains at most 9 points. Such 6-fold covers of \mathbf{P}^1 depend on at most $9 - 3 = 6$ parameters. Therefore, for a sufficiently general choice of C , the map ϕ will have at least 3 ramification points with the desired property. \square

We assume from now on that $[\partial]$ lies on no trisecants ($[\partial] \notin Q$) or tangents to C .

3.2. Case 2: $\ell_{\partial, \partial'} \cap C = \{p\}$. Here we mean that the line $\ell_{\partial, \partial'}$ is *not* tangent to C . In this case we have

$$\Gamma_{\text{red}} = X_3(g_5^1) \cup (X_2(g_5^1) + p) \subset C^{(3)}$$

where $g_5^1 \subset |\omega_C|$ is the base-point-free pencil cut on C by planes through $\ell_{\partial, \partial'}$. As before, we see that Γ is reduced. The involution

$$\tau : x + y + z \mapsto K_C - x - y - z$$

exchanges $X_3(g_5^1)$ and $X_2(g_5^1) + p$. A similar (simpler) calculation as before shows that $X_2(g_5^1)$, hence also $X_3(g_5^1)$, is irreducible. As the scheme $\Gamma \cap \partial \partial' \Theta$ is invariant under τ , we see that if it contains one component of Γ , it also contains the other. This is therefore not possible, hence this scheme is finite.

3.3. Case 3: $\ell_{\partial, \partial'} \cap C = \{p, q\}$. Here we mean that the line $\ell_{\partial, \partial'}$ intersects C in exactly two distinct points p and q .

Let ∂_p and ∂_q be elements of $T_{A,0}$ mapping to p and q respectively. Let W_p be the image in Θ of $p + C^{(2)} \subset C^{(3)}$. We have

$$\Theta \cap \partial_p \Theta = W_p \cup (K_C - W_p) = W_p \cup \tau(W_p).$$

Since $[\partial] \notin Q$, the linear system $|K_C - p - q|$ is a base-point-free g_4^1 and the curve $X_2(K_C - p - q)$ is irreducible as before. We have $\Gamma = \Theta \cap \partial \Theta \cap \partial' \Theta = \Theta \cap \partial_p \Theta \cap \partial_q \Theta$ and we check that this curve is reduced and has four irreducible components:

$$\begin{aligned} \Gamma_1 &= p + q + C & \Gamma_2 &= p + X_2(K_C - p - q) \\ \tau(\Gamma_1) &= X_3(K_C - p - q) & \tau(\Gamma_2) &= q + X_2(K_C - p - q). \end{aligned}$$

If $\Theta \cap \partial \Theta \cap \partial' \Theta \cap \partial \partial' \Theta$ contains a component of Γ , it also contains its image by τ . It will therefore be enough for our purpose to show that the section $\partial \partial' \theta$ of $\mathcal{O}_\Gamma(\Theta)$ vanishes identically neither on Γ_1 , nor on Γ_2 (both contained in W_p).

Let ι_{p+q} be the embedding $x \mapsto p + q + x$ of C into A , with image Γ_1 . We have $\iota_{p+q}^* \Theta \equiv K_C - p - q$. Let $p + p_1 + p_2 + p_3$ and $q + q_1 + q_2 + q_3$ be the divisors of $|K - p - q|$ containing p and q . For a sufficiently general choice of ∂ , these two divisors will be reduced and disjoint.

Lemma 4. *The section $\iota_{p+q}^* \partial_p \partial_q \theta$ vanishes identically and*

$$\begin{aligned} \operatorname{div}(\iota_{p+q}^* \partial_p^2 \theta) &= p + p_1 + p_2 + p_3 \\ \operatorname{div}(\iota_{p+q}^* \partial_q^2 \theta) &= q + q_1 + q_2 + q_3 \end{aligned}$$

Proof. Let $\lambda \in \mathbf{k}$ and set $\partial_\lambda = \lambda \partial_p + \partial_q$. The support of

$$\operatorname{div}(\partial_p \partial_\lambda \theta) = \Theta \cap \partial_p \Theta \cap \partial_q \Theta \cap \partial_p \partial_\lambda \Theta = \Theta \cap \partial_p \Theta \cap \partial_\lambda \Theta \cap \partial_p \partial_\lambda \Theta$$

is the set of points of $\Theta \cap \partial_p \Theta = W_p \cup (K_C - W_p)$ whose tangent space contains ∂_λ .

It contains $p + q + x$ if the line $\langle q, x \rangle$ contains $[\partial_\lambda]$. In particular, $\partial_p \partial_q \theta$ vanishes identically on $p + q + C$ and $\partial_p \partial_\lambda \theta(2p + q) = 0$ for all λ . This implies $\partial_p^2 \theta(2p + q) = 0$. Moreover, $\partial_p \partial_\lambda \theta(p + 2q) \neq 0$ if $\lambda \neq 0$. In particular, $\iota_{p+q}^* \partial_p^2 \theta$ is a nonzero section of $\mathcal{O}_C(K_C - p - q)$ that vanishes at p , hence the lemma. \square

Write $\partial = \lambda\partial_p + \mu\partial_q$ and $\partial' = \lambda'\partial_p + \mu'\partial_q$, so that

$$\partial\partial'\theta = \lambda\lambda'\partial_p^2\theta + (\lambda\mu' + \lambda'\mu)\partial_p\partial_q\theta + \mu\mu'\partial_q^2\theta.$$

Since $[\partial]$ is not on C , both λ and μ are not zero, hence $\partial\partial'\theta$ does not vanish identically on Γ_1 . We have

$$\begin{aligned}\Gamma_1 \cap \Gamma_2 &= \{p + q + q_1, p + q + q_2, p + q + q_3\} \\ \Gamma_1 \cap \tau(\Gamma_2) &= \{p + q + p_1, p + q + p_2, p + q + p_3\} \\ \tau(\Gamma_1) \cap \Gamma_2 &= \{\tau(p + q + p_1), \tau(p + q + p_2), \tau(p + q + p_3)\}\end{aligned}$$

The section $\partial_p\partial_q\theta$ does not vanish identically on Γ , hence does not vanish identically on Γ_2 . At $p + q + q_1$, both $\partial_p\partial_q\theta$ and $\partial_q^2\theta$ vanish, but $\partial_p^2\theta$ does not. At $\tau(p + q + p_1)$, both $\partial_p\partial_q\theta$ and $\partial_p^2\theta$ vanish, but $\partial_q^2\theta$ does not. It follows that the sections $\partial_p^2\theta|_{\Gamma_2}$, $\partial_p\partial_q\theta|_{\Gamma_2}$, and $\partial_q^2\theta|_{\Gamma_2}$ are linearly independent, hence $\partial\partial'\theta$ does not vanish identically on Γ_2 .

We have proved that in all cases, the zero set of $\partial\partial'\theta$ on Γ is finite. This completes the proof of Proposition 2. \square

4. THE SCHEME $\Theta \cap \partial\Theta \cap \partial^2\Theta$

The fiber of $\psi_{\partial}^{-1}([\partial])$ is one-dimensional, equal to $\Theta \cap \partial\Theta \cap \partial^2\Theta$. We now study this curve. Let p be a general point of C . As above, we see that $\Theta \cap \partial_p\Theta \cap \partial_p^2\Theta$ has three irreducible components whose reduced underlying curves are

$$\begin{aligned}\Gamma_1 &= 2p + C & \tau(\Gamma_1) &= X_3(K_C - 2p) \\ \Gamma_2 &= \tau(\Gamma_2) = p + X_2(K_C - 2p)\end{aligned}$$

and Γ_1 and $\tau(\Gamma_1)$ have multiplicity 1, whereas Γ_2 has multiplicity 2.

Lemma 5. *For ∂ general, $\Theta \cap \partial\Theta \cap \partial^2\Theta$ contains no translates of C .*

Proof. A translate of C is contained in Θ if and only if it is of the type $x + y + C$, with $x, y \in C$. It is contained in $\Theta \cap \partial\Theta$ if and only if for every $t \in C$, the plane $\langle x, y, t \rangle$ contains $[\partial]$. This is only possible if $[\partial]$ is on the line $\langle x, y \rangle$. For ∂ general, there are exactly six distinct secants to C that contain $[\partial]$, none of which is trisecant or tangent. So there are exactly six distinct translates, say $x_i + y_i + C$, for $i \in \{1, \dots, 6\}$, contained in $\Theta \cap \partial\Theta$. Since the set of secants to C is irreducible, if one of these translates is contained in $\Theta \cap \partial\Theta \cap \partial^2\Theta$ for ∂ general, they all are. This implies

$$\Theta \cap \partial\Theta \cap \partial^2\Theta = \bigcup_{i=1}^6 (x_i + y_i + C)$$

which is not possible since a general $\Theta \cap \partial\Theta \cap \partial^2\Theta$ has at most four irreducible components by the description of $\Theta \cap \partial_p\Theta \cap \partial_p^2\Theta$ above. \square

Since $4p$ is not contained in a plane in \mathbf{P}^3 , the curves Γ_1 and $\tau(\Gamma_1)$ defined above are disjoint. Therefore, it follows from Lemma 5 that if a

general $\Theta \cap \partial\Theta \cap \partial^2\Theta$ is nonintegral, it is of the form $\Gamma_0 \cup \tau(\Gamma_0)$, where Γ_0 is integral, with cohomology class $\frac{1}{2}[\Theta]^3$, and distinct from $\tau(\Gamma_0)$.

5. PROOF OF THEOREM 1

We keep the same assumptions and notation as before. Let a be general in $A = \text{Pic}^3(C)$. If for some nonzero ∂' , the scheme $\Theta \cap \Theta_a \cap \partial'\Theta \cap \partial'\Theta_a$ has dimension 1, it contains a curve Γ_a that is stable by the involution $x \mapsto a - x$. When a specializes to a general $[\partial]$, this involution specializes to τ and Γ_a must specialize as a set to $\Gamma_0 \cup \tau(\Gamma_0)$. Since this curve has the same cohomology class as the curve $\Theta \cap \Theta_a \cap \partial'\Theta$, this means that the section $\partial'\theta_a$ vanishes identically on the curve $\Theta \cap \Theta_a \cap \partial'\Theta$ and this is absurd.

It follows that ψ_a is finite, hence the cotangent bundle of $\Theta \cap \Theta_a$ is ample.

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