THE CHOW RING OF THE MODULI SPACE OF CURVES OF GENUS 5

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# Introduction

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus $g$ over an algebraically closed field (of characteristic different from 2 and 3) and let $\overline{\mathcal{M}}_g$ be its compactification by Deligne-Mumford stable curves. The Chow rings of $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ have attracted much attention:

In characteristic 0, D. Mumford (see [17], part I) defined an intersection product on the Chow group of $\overline{\mathcal{M}}_g$ with rational coefficients. He used the fact that $\overline{\mathcal{M}}_g$ is locally the quotient of a smooth variety by a finite group and globally the quotient of a Cohen-Macaulay variety by a finite group. Recently, E. Looijenga (see [15], Theorem) showed that, over $\mathbb{C}$, $\overline{\mathcal{M}}_g$ is globally the quotient of a smooth variety by a finite group. Then M. Pikaart and J. de Jong (see [19], Theorem 3.1.1) extended this result to positive characteristic. This provides a new and simpler way to define an intersection product on the Chow group of $\overline{\mathcal{M}}_g$ using the intersection product on the Chow group of the smooth variety: if $\overline{\mathcal{M}}_g$ is the quotient of $X$ by the action of the finite group $G$, then, by [10] (Examples 1.7.6 and 8.3.12), $A_*(\overline{\mathcal{M}}_g) = A_*(X)^G$ where $A_*$ is the Chow group with rational coefficients and $A_*(X)^G$ is the subring of invariants of $A_*(X)$ for the action of $G$.

The Chow ring $A_*(\overline{\mathcal{M}}_g)$ being defined, one would like to know more about it. For instance, one would like to write down naturally occuring elements, generators and relations, etc. Ideally, one would like a presentation and a multiplication table with a "nice" description of the generators. With the case of the grassmannians as a model, Mumford defined ([17], page 299) some classes in $A_*(\overline{\mathcal{M}}_g)$ called the tautological classes and wrote some relations between them, using, as a main tool, the Riemann-Roch theorem. He then showed that the Chow ring of $\overline{\mathcal{M}}_2$ is generated by classes coming from the boundary and wrote a complete set of relations between tautological and boundary classes together with a multiplication table.

Continuing this work, C. Faber showed that the Chow ring of $\overline{\mathcal{M}}_3$ is generated by $\lambda$ (one of Mumford's codimension one tautological classes) and boundary classes. He also wrote a complete set of relations between the tautological and boundary classes and showed that $A_*(\mathcal{M}_3) \cong \mathbb{Q}[\lambda]/(\lambda^2)$. He then showed that $A_*(\mathcal{M}_4) \cong \mathbb{Q}[\lambda]/(\lambda^3)$ and wrote a set of generators and relations for the codimension 1 and 2 Chow groups of $\overline{\mathcal{M}}_4$: these are again generated by tautological and boundary classes (see [9], the principal results of the paper are valid in positive characteristic $\neq 2, 3$, see the end of section 4 below).

C. Faber also wrote a set of generators for the codimension one Chow group of $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ (in characteristic zero, see [8]).

Let us assume for a moment that the base field is $\mathbb{C}$. We would like to mention some topological results. In [12] (see the Introduction and page 238), J. Harer showed that $H^2(\mathcal{M}_g, \mathbb{Z}) \cong \mathbb{Z}$ (for $g \geq 5$). This implies that the Picard group of $\mathcal{M}_g$ is also isomorphic to $\mathbb{Z}$ (for $g \geq 5$, for $g \leq 4$, the result is true up to torsion by the above-mentioned results of Mumford and Faber). Later, Arbarello and Cornalba (see [1]) showed that, if $\delta_0, \delta_1, \dots, \delta_{[g/2]}$ are the classes of the codimension

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one irreducible boundary components, then $\lambda, \delta_0, \delta_1, \dots, \delta_{[g/2]}$ freely generate the Picard group of the moduli functor of $\overline{\mathcal{M}}_g$. It follows, in particular, that $\lambda, \delta_0, \delta_1, \dots, \delta_{[g/2]}$ give a basis of $A_{3g-4}(\overline{\mathcal{M}}_g)$.

Then, in [11], Harer showed that $H^4(\mathcal{M}_g, \mathbb{Q})$ has dimension 2 when $g \geq 12$. D. Edidin [7] used this to give a basis for $H_{2(3g-3)-4}(\overline{\mathcal{M}}_g, \mathbb{Q})$ consisting of tautological and boundary classes for $g \geq 12$. More precisely, he showed the homological independence of two of the tautological classes ($\lambda^2$ and $\kappa_2$) together with the boundary classes for $g \geq 6$. Harer's result then permitted him to conclude that these were also generators for $g \geq 12$.

It was conjectured by Mumford (see [17] page 272) that if $k$ is small compared to $g$, then $H^{2k}(\mathcal{M}_g, \mathbb{Q})$ is generated by tautological classes and $H^{2k+1}(\mathcal{M}_g, \mathbb{Q}) = 0$ (by results of Harer, $H^k(\mathcal{M}_g, \mathbb{Q})$ is independent of $g$ for $g$ large enough).

In this paper we continue the study of Chow rings of moduli spaces of low genus curves. Specifically, we consider the case $g = 5$. We write down a geometric stratification of $\mathcal{M}_5$ such that the strata have trivial Chow groups. The classes of the closures of the strata will thus form a set of generators for the Chow ring of $\mathcal{M}_5$. We show that our generators can be expressed as polynomials in Mumford's tautological classes. It thus follows

**Theorem** The Chow ring of $\mathcal{M}_5$ (with coefficients in $\mathbb{Q}$) is generated by the tautological classes.

C. Faber indicated (private communication) that he has shown that in genus 5 the ring generated by the tautological classes is generated by $\lambda$ and the relation $\lambda^4 = 0$ holds in it. Combining our result with the result of Faber, we obtain that the Chow ring of $\mathcal{M}_5$ is a quotient of $\mathbb{Q}[\lambda]/(\lambda^4)$. We therefore have the following:

$$A_*(\mathcal{M}_2) \cong \mathbb{Q}$$
$$A_*(\mathcal{M}_3) \cong \mathbb{Q}[\lambda]/(\lambda^2)$$
$$A_*(\mathcal{M}_4) \cong \mathbb{Q}[\lambda]/(\lambda^3)$$
$$A_*(\mathcal{M}_5) \longleftarrow \mathbb{Q}[\lambda]/(\lambda^4)$$

with the last map onto. We do not know whether $\lambda^3$ is nonzero in $A_*(\mathcal{M}_5)$. Note that if one could show the existence of a complete 3-dimensional subvariety of $\mathcal{M}_5$, then it would follow that $\lambda^3$ is nonzero since $\lambda$ is ample on $\mathcal{M}_g$ (It has been shown by Diaz that the dimension of any complete subvariety of $\mathcal{M}_g$ is at most $g-2$ (see [5] Theorem 4 page 407, his proof works in case the characteristic is zero or greater than $g$).) Also note that, for $g \geq 6$, the Chow ring of $\mathcal{M}_g$ cannot be generated by $\lambda$ anymore (at least in characteristic zero) since, by Edidin's result, $A_{3g-5}(\mathcal{M}_g)$ has dimension at least 2.

The paper is organized as follows:

In the first section we introduce a preliminary stratification of $\mathcal{M}_5$ into well-known subvarieties ($U$, $B$, $T$ and $H$) and reduce the proof of our theorem (roughly) to showing that the Chow rings of $U$, $V := U \cup B$ and $W := U \cup T \cup B$ are generated by tautological classes. In the second section we gather some preliminary results about quartic Del Pezzo surfaces and canonical curves of genus 5. In the third section we show that the Chow ring of the "biggest" stratum $U$ of our preliminary stratification is generated by tautological classes. In the last two sections we show that the Chow rings of $V$ and $W$ are generated by tautological classes.

**Acknowledgments:** Thanks are due to E. Arbarello: it was after a stimulating conversation with him that I became seriously interested in studying Chow rings of moduli spaces of curves. Thanks are also due to C. Faber for much of this work was inspired by his beautiful paper "Chow rings of moduli spaces of curves" ([9]). This work was begun while the author was a research fellow at the Mathematical Sciences Research Institute (MSRI) at Berkeley and supported by MSRI and the NSF grant number DMS-9204266.

## Notation and Conventions

We denote by $\omega_X$ the canonical sheaf of a smooth curve $X$ of genus 5 and let $K_X$ be an arbitrary element of the linear system $|\omega_X|$. We let $g^r_n$ denote an arbitrary complete linear system of degree $n$ and (projective) dimension $r$ on $X$. We identify a non-hyperelliptic curve $X$ with its canonical model. For a divisor $D$ on $X$, we will denote by $\langle D\rangle$ its linear span in the canonical space of $X$. For two points $s$ and $t$ on $X$, we will denote by $\pi_{st} : |K_X|^* \cong \mathbb{P}^4 \longrightarrow |K_X - s - t|^* \cong \mathbb{P}^2$ the projection from $\langle s+t\rangle$ and by $X^{st} \subset \mathbb{P}^2$ the image of $X$ by $\pi_{st}$. By a bielliptic curve we mean a curve which is a (ramified) double cover of an elliptic curve.

We will call the point of $\mathcal{M}_g$ corresponding to $X$ "the moduli point of $X$" and we will denote this point by $m_X$.

Finally, by the Chow ring or Chow group $A_*(M)$ of a scheme $M$ we always mean the Chow ring or Chow group with coefficients in $\mathbb{Q}$. For a subscheme $P$ of $M$, we will denote by $[P]_M$ the (usual fundamental) class of $P$ in $A_*(M)$. By a node of $M$ we mean an ordinary double point.

### 1. A preliminary stratification

We will first write $\mathcal{M}_5$ as a disjoint union of well-known locally closed subvarieties:

$H$: closed subvariety of $\mathcal{M}_5$ parametrizing hyperelliptic curves.
$T$: locally closed subvariety of $\mathcal{M}_5$ parametrizing trigonal (non-hyperelliptic) curves.
$B$: closed subvariety of $\mathcal{M}_5$ parametrizing bielliptic curves.
$U$: $= \mathcal{M}_5 \setminus (T \cup B \cup H)$

Note that the closure of $T$ in $\mathcal{M}_5$ is the union of $T$ and $H$ and does not intersect $B$ and that a curve with moduli point in $T$ has a unique $g^1_3$ (see [2] page 366 Exercises C-1 and C-2, the result holds in characteristic $\neq 2,3$ since for us $d_1, d_2 = 2$ or 3). By [10] page 21, for any closed subscheme $Z$ of a scheme $Y$ and for any $k \in \mathbb{Z}$ the sequence

$$A_k(Z) \xrightarrow{i_*} A_k(Y) \xrightarrow{j^*} A_k(Y \setminus Z) \longrightarrow 0$$

where $i$ is the embedding $Z \hookrightarrow Y$ and $j$ is the embedding $(Y \setminus Z) \hookrightarrow Z$, is exact. Therefore to prove our theorem it suffices to show that

1. $A_*(U)$ is generated by tautological classes
2. the image of $A_*(B)$ in $A_*(U \cup B)$ is generated by tautological classes
3. the image of $A_*(T)$ in $A_*(U \cup T \cup B)$ is generated by tautological classes
4. the image of $A_*(H)$ in $A_*(\mathcal{M}_5)$ is generated by tautological classes

Note that Mumford showed in [17] page 314 that $[H]_{\mathcal{M}_5}$ is a combination of tautological classes. Also, it is shown in [9] Lemma 1.4 that $A_*(H)$ is trivial. So the last step has already been done. (The proofs remain valid in characteristic $\neq 2$.)

To prove the other steps, we will stratify each of $U, B$ and $T$ separately. We will start with $U$ and $B$. The stratification is based on the fact that for a given quartic Del Pezzo surface (with rational double points and determined up to projecitve equivalence) there is a locally closed subset of $V = U \cup B$ parametrizing curves whose canonical model embeds into the surface. For these two stratifications we need some preliminaries which we gather in the next section.

## 2. Quartic Del Pezzo surfaces and non-trigonal curves of genus 5

The canonical model of a smooth non-trigonal (and non-hyperelliptic) curve $X$ of genus 5 is the base locus of a net of quadrics in $\mathbb{P}^4$. Let $\Pi \cong \mathbb{P}^2$ parametrize the net of quadrics containing $X$. Let $Q \subset \Pi$ be the plane quintic parametrizing the singular quadrics. It is well-known that $Q$ has at worst ordinary double points as singularities and that these singularities correspond exactly to the quadrics of rank 3 containing $X$ (cf. [3], pages 321 and 361-362). Let $l$ be a pencil of quadrics containing $X$. Suppose that $l$ is not contained in $Q$ and let $q_1, ..., q_5$ be the points of intersection of $Q$ with $l$. Then we have

**Lemma 2.1.** *Suppose that $q_1, ..., q_5$ are distinct. Then the base locus of $l$ is a smooth quartic Del Pezzo surface $S(l)$. The 16 lines in $S(l)$ are all secant to $X$. Let $\langle s+t \rangle$ be one of the lines in $S(l)$, with $s, t \in X$. Then $X^{st}$ is a plane sextic with 5 distinct double points, say $p_1, ..., p_5$. The points $p_1, ..., p_5$ are at worst ordinary cusps and no three of them are on a line. The conic through $p_1, ..., p_5$ also contains $\overline{s} := \pi_{st}(s)$ and $\overline{t} := \pi_{st}(t)$. The double point $p_i$ is the image of the vertex $s_i$ of $q_i$. The surface $S(l)$ is isomorphic to the blow up of $\mathbb{P}^2$ at the points $p_1, ..., p_5$.*

*Proof:* If the base locus $S(l)$ of $l$ has a singular point $p$, then there is an element of $l$, say $q$, which is singular at $p$. Since $l$ intersects $Q$ in 5 distinct points, all these points are smooth points of $Q$ hence correspond to quadrics of rank 4. So $q$ has rank 4. In the space parametrizing the quadrics in $\mathbb{P}^4$ (isomorphic to $\mathbb{P}^{14}$), the locus of quadrics of rank $\leq 4$ is an irreducible hypersurface $G$ of degree 5 (defined by the vanishing of the determinant of the 5 by 5 matrix associated to a quadric). An elementary computation shows that the projectivized Zariski tangent space to $G$ at $q$ is the set of quadrics containing the vertex of $q$. Hence $l$ is tangent to $G$ at $q$. It follows that the number of singular quadrics in $l$ is less than 5 which contradicts the assumption.

So the base locus of $l$ is smooth and it is well-known in that case that it is a quartic Del Pezzo surface and contains exactly 16 distinct lines which are all secant to $X$ since $X$ is the complete intersection of $S(l)$ with a quadric. Since, by the above, none of the $s_i$'s is on $S(l)$, for each $i$, the line $\langle s+t \rangle$ does not contain $s_i$. So $\langle s+t \rangle$ is contained in exactly one ruling of $q_i$ which cuts a $g^1_4$, say $g_i$, on $X$ such that $h^0(g_i - s - t) > 0$ (see e.g. [16] pages 342-343). Conversely, any $g^1_4$ such that $h^0(g^1_4 - s - t) > 0$ is cut by a ruling of a quadric $q$ of rank $\leq 4$ which must contain $S(l)$ since its intersection with $S(l)$ contains $X \cup \langle s+t \rangle$ (curve of degree $9 > 2.4 = \deg(q).\deg(S(l))$). Hence there are exactly 5 distinct lines $\langle p'_i + p''_i \rangle$ ($p'_i + p''_i \equiv g_i - s - t$) in $S(l)$ intersecting $\langle s+t \rangle$ and they project to five distinct double points $p_1, ..., p_5$ of $X^{st}$. The point $s_i$ belongs to the plane $\langle s + t + p'_i + p''_i \rangle$ so that $p_i$ is the image of $s_i$ by the projection $\pi_{st} : \mathbb{P}^4 \longrightarrow \mathbb{P}^2$ from $\langle s+t \rangle$.

It is an easy exercise in intersection theory to check that the restriction of $\pi_{st}$ to $S(l)$ is an isomorphism on the complement of the lines $\langle p'_i + p''_i \rangle$ (computation of the residual intersection of a plane containing $\langle s+t \rangle$ with $S(l)$, see e.g. [10] Example 9.1.1). Therefore the surface $S(l)$ is isomorphic to the blow up of $\mathbb{P}^2$ at $p_1, ..., p_5$ and, by adjunction, it is the image of this blow up in $\mathbb{P}^4$ by the linear system of strict transforms of cubics through $p_1, ..., p_5$. Hence, no three of the points $p_1, ..., p_5$ are collinear because $S(l)$ is smooth. Finally, $s + t \neq p'_i + p''_i$ for all $i$ because this is the case for $X, s, t$ general so that if for some $X, s, t$, $s + t = p'_i + p''_i$ for some $i$, then $S(l)$ can't be smooth because the number of lines in $S(l)$ would be less than 16. Finally, by adjunction, $s + t \equiv 2(K_X - s - t) - \sum_{i=1}^{5} p'_i + p''_i$. Therefore $p_1, ..., p_5, \overline{s}$ and $\overline{t}$ lie on a conic. Q.E.D.

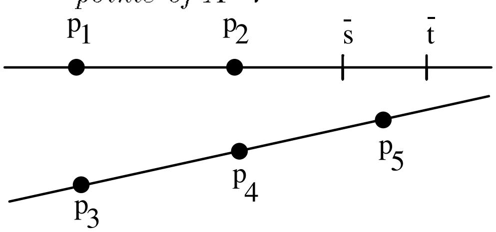
We now proceed to determine $S(l)$ for each $l \subset \Pi$ which is neither contained in $Q$ nor transverse to it. In what follows, we call $p_i$ an infinitely near double point of $X^{st}$ if $p_i$ is a double point of the strict transform of $X^{st}$ in some blow up $\widetilde{\mathbb{P}}^2$ of $\mathbb{P}^2 \supset X^{st}$. We say that $p_j$ is infinitely near of order 1 to $p_i$ if $p_j$ is an element of the exceptional divisor of the blow up of $\widetilde{\mathbb{P}}^2$ at $p_i$. In the set of double points of $X^{st}$ we include those that are infinitely near. In this way we always have 5 distinct double points for $X^{st}$ (note that the morphism $X \longrightarrow X^{st}$ is birational because $l$ is not contained in $Q$). The list of anti-canonical Del Pezzo surfaces that we present in the lemma below is the same as that in [4] page 38. We need to present these from a different point of view for our purposes. The essence of the lemma is that the data of the configuration of double points of $X^{st}$ is equivalent to the singularities and number of lines of $S(l)$ which is in turn equivalent to the data of the "type" of $l \cap Q$, where "type" is defined as follows

**Definition 2.2.** *To each point of intersection of $l$ with $Q$ we can associate two numbers: the first is the multiplicity of that point on $Q$ and the second the order of contact of $l$ with $Q$ at that point. In this way we associate to $l \cap Q$ an unordered sequence of ordered pairs of positive integers of length $\leq 5$. We call this sequence the* type *of $l \cap Q$ and denote it by* type$(l \cap Q)$.

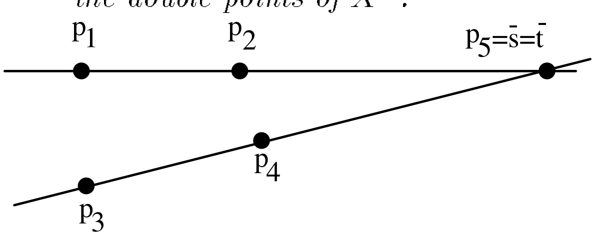
For instance, type$(l \cap Q) = \{(1,1),(1,1),(1,1),(1,1),(1,1)\}$ if and only if $S(l)$ is smooth.

**Lemma 2.3.** *Suppose that $l \not\subset Q$ and that $q_1, \dots, q_5$ are* not *distinct. Whenever $S(l)$ has a double point we can and will assume that $\langle s+t \rangle$ contains it. If $S(l)$ has two double points (or more) and contains the line through them we can and will assume that $\langle s+t \rangle$ is that line. Then, up to a permutation of the $q_i$'s, we have the following possibilities (in the following, the conic through $\overline{s}, \overline{t}, p_1, ..., p_5$ (see 2.1) has degenerated to the union of two lines or to twice a line: these are the lines represented on the pictures):*

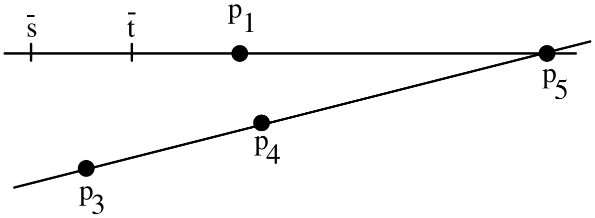
1. type$(l \cap Q) = \{(1,2),(1,1),(1,1),(1,1)\}$
   *The line $l$ is simply tangent to $Q$ at $q_1 = q_2$ and is transverse to $Q$ at $q_3, q_4, q_5$. This is equivalent to: $S(l)$ has only one node (i.e., an ordinary double point) and contains 12 lines (4 of which pass through its node). Equivalently, we have the following picture for the double points of $X^{st}$:*




2. type$(l \cap Q) = \{(1,2),(1,2),(1,1)\}$
   *The line $l$ is simply tangent to $Q$ at two smooth points $q_1 = q_2$ and $q_3 = q_4$. This is equivalent to: $S(l)$ has two nodes and contains 9 lines. It is also equivalent to the following picture for the double points of $X^{st}$.*



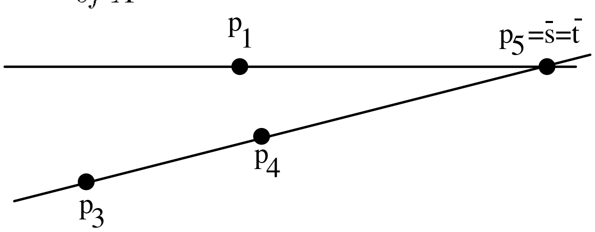

   *In this case $S(l)$ contains the line through its nodes and there are 2 other lines in $S(l)$ through each of the nodes.*
3. type$(l \cap Q) = \{(1,3),(1,1),(1,1)\}$
   *The line $l$ has contact of order 3 with $Q$ at the smooth point $q_3 = q_4 = q_5$ and meets $Q$ transversely at $q_1 \neq q_2$. This is equivalent to: $S(l)$ has one double point of type $A_2$ and contains 8 lines. It is also equivalent to the following picture for the double points of $X^{st}$*




*with $p_2$ infinitely near of order 1 to $p_5$. In this case 4 of the lines in $S(l)$ pass through its double point.*

4. $\text{type}(l \cap Q) = \{(1,3),(1,2)\}$

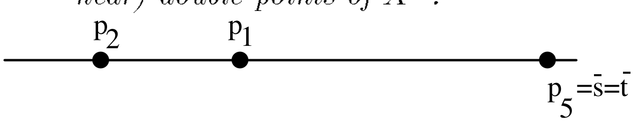
*The line $l$ has contact of order 3 with $Q$ at the smooth point $q_3 = q_4 = q_5$ and is tangent to $Q$ at the smooth point $q_1 = q_2 \neq q_3$. Equivalently, $S(l)$ has one node and one double point of type $A_2$ and contains 6 lines. Equivalently, we have the following picture for the double points of $X^{st}$*




*where $p_2$ is infinitely near of order 1 to $p_5$ and the line $\langle p_1 + p_5 \rangle$ is tangent to the branch(es) of $X^{st}$ at $p_5$. In this case one of the lines in $S(l)$ passes through the two double points, one other passes through the node and two others through the $A_2$-double point.*

5. $\text{type}(l \cap Q) = \{(1,4),(1,1)\}$

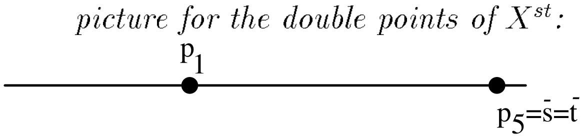
*The line $l$ has contact of order 4 with $Q$ at the smooth point $q_1 = q_2 = q_3 = q_4 (\neq q_5)$. Equivalently, the surface $S(l)$ has one double point of type $A_3$ and contains 5 lines. Equivalently, we can choose $\langle s + t \rangle$ in such a way that we have the following picture for the (non-infinitely near) double points of $X^{st}$:*




*with $p_3$ infinitely near of order 1 to $p_1$ and $p_4$ infinitely near of order 1 to $p_2$. In this case 3 of the lines in $S(l)$ pass through the double point.*

6. $\text{type}(l \cap Q) = \{(1,5)\}$

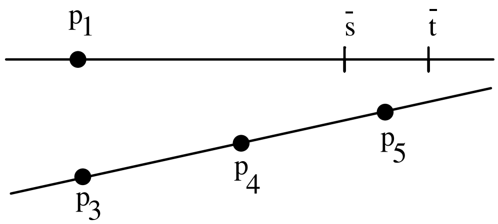
*The line $l$ has contact of order 5 with $Q$ at the smooth point $q_1 = q_2 = q_3 = q_4 = q_5$. Equivalently, $S(l)$ has one double point of type $A_4$ and contains 3 lines (2 of which pass through the double point). Equivalently, we can choose $\langle s + t \rangle$ in such a way that we have the following picture for the double points of $X^{st}$:*




*with $p_3$ infinitely near of order 1 to $p_1$, $p_4$ infinitely near of order 1 to $p_2$, $p_2$ infinitely near of order 1 to $p_5$ and the line $\langle p_1 + p_5 \rangle$ tangent to the branch(es) of $X^{st}$ at $p_5$.*

7. $\text{type}(l \cap Q) = \{(2,2),(1,1),(1,1),(1,1)\}$

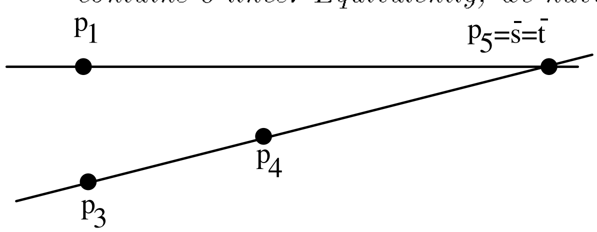
*The line $l$ contains a node $q_1 = q_2$ of $Q$, is not tangent to any of the branches of $Q$ at $q_1$ and is elsewhere transverse to $Q$. Equivalently, $S(l)$ has two nodes and contains 8 lines. Equivalently, we have the following picture for the double points of $X^{st}$:*




*with* $p_2$ *infinitely near of order* 1 *to* $p_1$ *and the line* $\langle p_1 + \overline{s}\rangle$ *tangent to the branch(es) of* $X^{st}$ *at* $p_1$*. In this case the nodes of* $S(l)$ *are located on the singular locus* $Sing(q_1)$ *of* $q_1$ *which is not contained in* $S(l)$ *and* 4 *lines (in* $S(l)$*) pass through each node.*

8. $\text{type}(l \cap Q) = \{(2,2),(1,2),(1,1)\}$

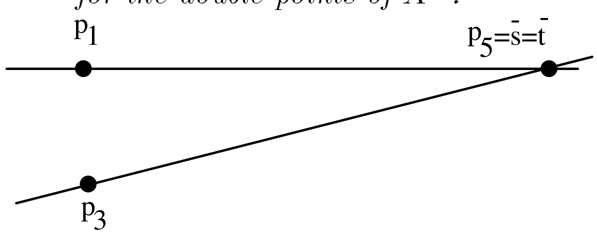
*The line* $l$ *contains a node* $q_1 = q_2$ *of* $Q$*, is not tangent to any of the branches of* $Q$ *at* $q_1$ *and is simply tangent to* $Q$ *at a smooth point* $q_3 = q_4$*. Equivalently,* $S(l)$ *has three nodes and contains* 6 *lines. Equivalently, we have the following picture for the double points of* $X^{st}$*:*




*with* $p_2$ *infinitely near of order* 1 *to* $p_1$ *and the line* $\langle p_1 + p_5\rangle$ *tangent to the branch(es) of* $X^{st}$ *at* $p_1$*. In this case two of the nodes of* $S(l)$*, say* $o_1$ *and* $o_2$*, are on* $Sing(q_1)$ *and the other is* $Sing(q_3)$*. The surface* $S(l)$ *contains the lines* $\langle o_i, Sing(q_3)\rangle$ *but does not contain* $Sing(q_1)(=\langle o_1, o_2\rangle)$*.*

9. $\text{type}(l \cap Q) = \{(2,2),(1,3)\}$

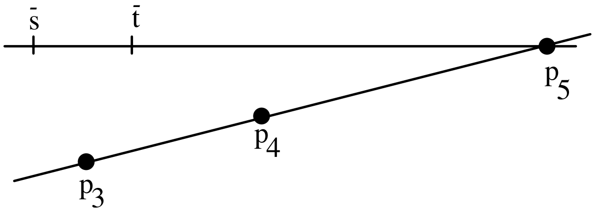
*The line* $l$ *contains a node* $q_1 = q_2$ *and has contact of order* 3 *with* $Q$ *at a smooth point* $q_3 = q_4 = q_5$*. Equivalently,* $S(l)$ *has two nodes and one double point of type* $A_2$ *and contains* 4 *lines (*2 *of which, as before, are* $\langle o_i, Sing(q_3)\rangle$*). Equivalently, we have the following picture for the double points of* $X^{st}$*:*




*with* $p_2$ *infinitely near of order* 1 *to* $p_1$*,* $p_4$ *infinitely near of order* 1 *to* $p_5$*, the line* $\langle p_1 + p_5\rangle$ *tangent to the branch(es) of* $X^{st}$ *at* $p_1$ *and the line* $\langle p_3 + p_5\rangle$ *tangent to the branch(es) of* $X^{st}$ *at* $p_5$*.*

10. $\text{type}(l \cap Q) = \{(2,3),(1,1),(1,1)\}$

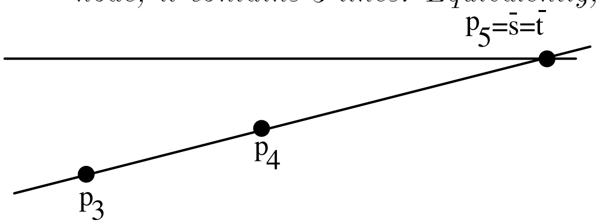
*The line* $l$ *contains a node* $q_1 = q_2$*, is simply tangent to one of the branches of* $Q$ *at* $q_1$ *and is elsewhere transverse to* $Q$*. Equivalently,* $S(l)$ *has one double point of type* $A_3$ *and contains* 4 *lines (all through the double point). Equivalently, the picture for the double points of* $X^{st}$ *is as follows:*




*with $p_2$ infinitely near of order 1 to $p_1$, $p_1$ infinitely near of order 1 to $p_5$ and the line $\langle p_5 + \overline{s}\rangle$ tangent to the branch(es) of $X^{st}$ at $p_5$.*

11. $\mathrm{type}(l \cap Q) = \{(2,3),(1,2)\}$

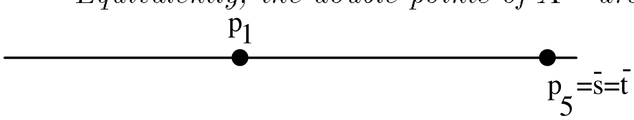
*The line $l$ contains a node $q_1 = q_2$, is simply tangent to one of the branches of $Q$ at $q_1$ and is tangent to $Q$ at a smooth point. Equivalently, $S(l)$ has one double point of type $A_3$ and one node; it contains 3 lines. Equivalently, the double points of $X^{st}$ are as follows:*




*with $p_2$ infinitely near of order 1 to $p_1$ and $p_1$ infinitely near of order 1 to $p_5$. The horizontal line in the picture is tangent to the branch(es) of $X^{st}$ at $p_1$ and it intersects $X^{st}$ only at $p_1$.*

12. $\mathrm{type}(l \cap Q) = \{(2,4),(1,1)\}$

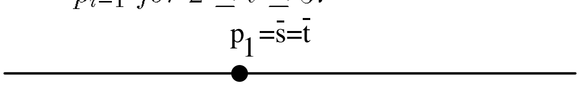
*The line $l$ contains a node $q_1 = q_2$ and has contact of order 3 with one of the branches of $Q$ at $q_1$. This is equivalent to: $S(l)$ has one double point of type $D_4$ and contains 2 lines. Equivalently, the double points of $X^{st}$ are:*




*with $p_4$ infinitely near of order 1 to $p_3$, $p_3$ infinitely near of order 1 to $p_2$ and $p_2$ infinitely near of order 1 to $p_1$. The line $\langle p_1 + p_5\rangle$ is tangent to the branch(es) of $X^{st}$ at $p_1$.*

13. $\mathrm{type}(l \cap Q) = \{(2,5)\}$

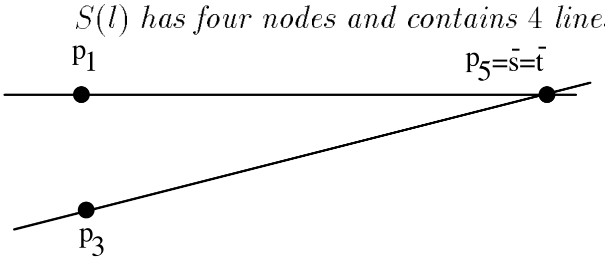
*The line $l$ contains a node of $Q$ and has contact of order 4 with one of the branches of $Q$ there. Equivalently, $S(l)$ has one double point of type $D_5$ and contains 1 line. Equivalently, $X^{st}$ has only one non-infinitely near double point, the point $p_i$ is infinitely near of order 1 to $p_{i-1}$ for $2 \leq i \leq 5$:*




*The line in the picture is tangent to the branch(es) of $X^{st}$ at $p_1$ and it intersects $X^{st}$ only at $p_1$.*

14. $\mathrm{type}(l \cap Q) = \{(2,2),(2,2),(1,1)\}$

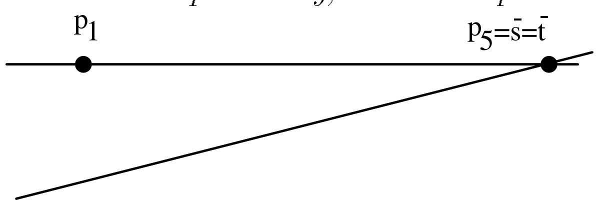
*The line $l$ passes through two nodes of $Q$ and contains a smooth point of $Q$. Equivalently, $S(l)$ has four nodes and contains 4 lines. Equivalently, the double points of $X^{st}$ are as follows:*



*with* $p_2$ *infinitely near of order* 1 *to* $p_1$ *and* $p_4$ *infinitely near of order* 1 *to* $p_3$. *The line* $\langle p_i + p_5\rangle$ *is tangent to the branch(es) of* $X^{st}$ *at* $p_i$ *for* $i = 1$ *or* 3.

15. type$(l \cap Q) = \{(2,3),(2,2)\}$

*The line* $l$ *passes through two nodes of* $Q$ *and is tangent to one of the branches of* $Q$ *at one of the nodes. Equivalently,* $S(l)$ *has two nodes and one double point of type* $A_3$*; it contains* 2 *lines. Equivalently, the double points of* $X^{st}$ *are as in the picture:*




*with* $p_4$ *infinitely near of order* 1 *to* $p_3$, $p_3$ *infinitely near of order* 1 *to* $p_5$ *and* $p_2$ *infinitely near of order* 1 *to* $p_1$. *The line* $\langle p_1 + p_5\rangle$ *is tangent to the branch(es) of* $X^{st}$ *at* $p_1$. *The slanted line in the picture is tangent to the branch(es) of* $X^{st}$ *at* $p_5$ *and it intersects* $X^{st}$ *only at* $p_5$.

*Proof:* The surface $S(l)$ is irreducible and generically reduced: otherwise $X$ must be contained in a plane, a quadric surface or a cubic surface in $\mathbb{P}^4$. The first two cases are impossible because $X$ is nondegenerate. The third case is excluded because $X$ is neither trigonal nor hyperelliptic: By [18] page 365, any irreducible nondegenerate cubic surface must be a rational normal scroll in which case $X$ is either hyperelliptic or trigonal since, by Riemann-Roch, the ruling of the rational normal scroll cuts either a $g^1_2$ or a $g^1_3$ or a $g^2_4$ on $X$. Since $S(l)$ is a complete intersection, it is Cohen-Macaulay. Since $X$ is the complete intersection of $S(l)$ with a quadric and $X$ is smooth, the surface $S(l)$ is smooth in codimension 1 and therefore normal.

Therefore, by, for instance, [18] Theorem 8 pages 366-367, (see also [4] page 34) the surface $S(l)$ is either the projection of a rational normal quartic scroll in $\mathbb{P}^5$ or an anticanonical Del Pezzo surface or a cone over a normal elliptic curve in $\mathbb{P}^3$. The first case is ruled out as before since the pencil of lines in $S(l)$ would cut a $g^1_2$ or a $g^1_3$ or a $g^2_4$ on $X$. In the third case every quadric of $l$ is singular at the vertex of $S(l)$, hence is of rank at most 4 and $l \subset Q$: this contradicts our hypothesis. Therefore $S(l)$ is an anti-canonical Del Pezzo surface of degree 4. It is the image in $\mathbb{P}^4$ of the blow up of $\mathbb{P}^2$ at the double points (including the infinitely near ones) of $X^{st}$ by the linear system of strict transforms of cubics passing through these double points (this follows, for instance, from the fact that the image of the strict transform of $X^{st}$ in $\mathbb{P}^4$ is $X$ which is smooth).

We will now show that $S(l)$ has one double point for each quadric of rank 4 appearing with multiplicity 2 in $l \cap Q$ and one or two double points for each quadric of rank 3 in $l$ (note that we already saw in Lemma 2.1 that $S(l)$ is smooth if $l$ contains 5 distinct quadrics of rank 4, i.e., if $l$ is transverse to the locus of quadrics of rank $\leq 4$):

If $l$ is tangent to $Q$ at a quadric $q_1$ of rank 4, then it is easily seen that all the quadrics of $l$ contain the vertex $s_1$ of $q_1$ and hence $S(l)$ has a double point (since $S(l)$ does not have points of multiplicity $\geq 3$) at $s_1$.

If $l$ contains a quadric $q$ of rank 3, then the only quadric of $\Pi$ containing $Sing(q)$ is $q$: Containing $Sing(q)$ is a linear condition and if two distinct quadrics of $l$ contain $Sing(q)$ then this is true for all the quadrics of $l$. If this is the case, then $S(l)$ is singular along $Sing(q)$. However, we saw above that $S(l)$ is normal. All the quadrics of $l \setminus \{q\}$ have the same intersection with $Sing(q)$ which is then either 2 distinct points or 1 point counted with multiplicity 2. It follows that $S(l)$ has 2 or 1 double points on $Sing(q)$.

To prove the beginning of the lemma, we show that every double point of $S(l)$ is on a secant of $X$. Since any double point is on the singular locus of some quadric parametrized by $Q$, we show that these singular loci are contained in the secant variety of $X$.

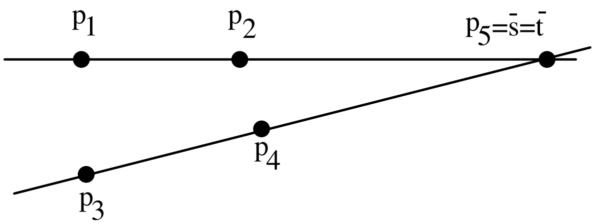
Let $s : Q \longrightarrow |K_X|^*$ be the rational map which sends a quadric $q \in Q$ of rank 4 to its vertex. Let $N$ be the normalization of $Q$ and $s_N : N \longrightarrow |K_X|^*$ the morphism induced by $s$. Let $s_N(N)$ be the image of $N$ by this morphism.

Suppose for a moment that $X$ is general. Then a general tangent line $l$ to $Q$ is also a general tangent line to $G$ (the hypersurface parametrizing singular quadrics in $|\mathcal{O}_{\mathbb{P}^4}(2)|$, see the proof of 2.1). It is then easily seen that $S(l)$ has one node and contains 12 lines, 4 of which contain the node of $S(l)$. Since all the lines in $S(l)$ are secant to $X$ ($X$ is the complete intersection of $S(l)$ with a quadric), it follows that $s_N(N)$ is contained in the secant variety of $X$ when $X$ is general and hence for all $X$.

Now we show that the singular loci of the quadrics of rank 3 containing $X$ are contained in the secant variety of $X$. Suppose for a moment that $X$ is a general curve with a vanishing theta-null (this means that $Q$ is a general plane quintic with a node). Then take $l$ to be a pencil of quadrics of $\Pi$ containing the quadric $q$ of rank 3 and general for this property. Then $l$ is also a line in $|\mathcal{O}_{\mathbb{P}^4}(2)|$ which intersects the subvariety parametrizing quadrics of rank $\leq 3$ and $l$ is general for this property. It is easily seen that $S(l)$ has 2 nodes which are the intersection of $S(l)$ with $Sing(q)$ and contains 4 lines through each node. Again, all the lines in $S(l)$ are secant to $X$. So it is enough to show that the two nodes of $S(l)$ move on $Sing(q)$ as $l$ varies. The two nodes are the intersection with $Sing(q)$ of any quadric $q'$ of $l$ distinct from $q$. So their sum is the divisor of degree 2 associated to $q'$ by the restriction map $\Pi \longrightarrow |\mathcal{O}_{Sing(q)}(2)|$. Hence it is sufficient to observe that $\Pi$ induces a pencil on $Sing(q)$: by [3] page 364, the projectivized tangent cone to $Q$ at $q$ is the union of the two pencils of quadrics tangent to $Sing(q)$ at one of the two elements $o_1, o_2$ of $s_N(N) \cap Sing(q)$ so that $\Pi|_{Sing(q)}$ contains the two divisors $2o_1, 2o_2$ (so the pencil is in fact *base point free*!). So the singular locus of a quadric of rank 3 in $\Pi$ is contained in the secant variety of $X$ for $X$ general with a vanishing theta-null and hence for all $X$ with vanishing theta-nulls because this is a closed condition.

In the case by case analysis below we will prove that the type of $l \cap Q$ determines the configuration of double points of $X^{st}$. The assertions about the singularities of $S(l)$ and the number and configuration of lines in $S(l)$ are easily seen to be equivalent to those about the configuration of the double points of $X^{st}$ with our choices of $\langle s+t \rangle$ by using the fact that $S(l)$ is the image of the blow up of $\mathbb{P}^2$ at the double points (including the infinitely near ones) of $X^{st}$ by the linear system of strict transforms of cubics passing through these double points. Since for different types of $l \cap Q$, we obtain different kinds of singularities and numbers of lines for $S(l)$, we obtain that the kind of singularities and number of lines in $S(l)$ determines the type of $l \cap Q$.

1. In this case, $q_1 = q_2$ has rank 4 and (we are supposing) $s_1 = s_2 \in \langle s+t \rangle$. Hence $\langle s+t \rangle$ is contained in both rulings of $q_1$ and $2(s+t) + p_1' + p_1'' + p_2' + p_2'' \in |\omega_X|$. It follows that $p_1' + p_1'' \neq p_2' + p_2''$ because $q_1$ has rank 4 and its rulings cut two distinct $g^1_4$ on $X$. So $p_1$ and $p_2$ are not infinitely near double points for $X^{st}$ and $p_1, p_2, \overline{s}$ and $\overline{t}$ are collinear. Since $\overline{s}, \overline{t}$ and the $p_i$'s are on a conic (see Lemma 2.1), $p_3, p_4, p_5$ are also collinear. Since $q_1, q_3, q_4, q_5$ are distinct, it follows that the $p_i$'s are not infinitely near to each other. The configuration of the points $p_1, \dots, p_5, \overline{s}, \overline{t}$ is as asserted unless $s + t = p_j' + p_j''$ for some $j$. If this is the case, then, since $X^{st}$ does not have infinitely near double points, we have $j = 3$ or $4$ or $5$, say $j = 5$. We would then have the following picture for the double points of $X^{st}$ and $\overline{s}, \overline{t}$:




However, this would imply that $S(l)$ has 2 double points which is not possible by the preliminary discussion above.

An observation which will be useful below is that the 4 lines $\langle s+t\rangle$, $\langle p_3'+p_3''\rangle$, $\langle p_4'+p_4''\rangle$, $\langle p_5'+p_5''\rangle$ all pass through $s_1$ (which is the node of $S(l)$). This is because

- these lines intersect in a point because they intersect two by two (and no three of them are coplanar): The line $\langle s+t\rangle$ intersects the others since it is coplanar with each of them. Since $p_3, p_4, p_5$ are collinear, we have $s+t+p_3'+p_3''+p_4'+p_4''+p_5'+p_5'' \in |\omega_X|$. Hence, by Riemann Roch, since $|s+t+p_i'+p_i''|$ is a $g^1_4$, so is $|p_j'+p_j''+p_k'+p_k''|$ with $\{i,j,k\}=\{3,4,5\}$.
- the point of intersection of the four lines is singular on $S(l)$ since the projectivized Zariski tangent space to $S(l)$ at this point contains all four lines.

2. In this case $s_1 = s_2$ and $s_3 = s_4$. We first claim that we can suppose $s_1, s_3 \in \langle s+t\rangle$. We already know that we can suppose $s_1 \in \langle s+t\rangle$. If $s_3 \notin \langle s+t\rangle$, then $\langle s+t\rangle$ is contained in only one ruling of $q_3$ and $p_4'+p_4'' = p_3'+p_3''$. Then $s+t+2p_3'+2p_3''+p_5'+p_5'' \in |\omega_X|$ and $s_3 \in \langle p_3'+p_3''\rangle$. As this case is a limit of case 1, the line $\langle p_3'+p_3''\rangle$ contains $s_1$. So we can replace $s+t$ by $p_3'+p_3''$ and suppose that $s_1, s_3 \in \langle s+t\rangle$.

   Since $\langle s+t\rangle$ contains $s_1$ and $s_3$, it still intersects 5 *distinct* secants to $X$ (see the proof of case 1). So the $p_i$'s are not infinitely near to each other. Now, $\bar{s}$ and $\bar{t}$ are on the two *distinct* lines $\langle p_1, p_2\rangle$ and $\langle p_3, p_4\rangle$. Hence $\bar{s}=\bar{t}=p_5$ since any line cuts a divisor of degree 6 on $X^{st}$.
3. In this case we see that $\langle s+t\rangle$ intersects exactly 4 distinct secants to $X$ (since $\langle s+t\rangle \ni s_1$, see the proof of case 1). So $X^{st}$ has four non-infinitely near double points, say $p_1, p_3, p_4, p_5$ and, $p_2$ is, for instance, infinitely near to $p_5$ (which is then either a tacnode or a cusp of higher order of $X^{st}$). Since this is a degeneration of case 1, we see that $p_3, p_4, p_5$ are collinear as well as $\bar{s}, \bar{t}, p_1, p_5$. Since $X^{st}$ has degree 6, neither of the two lines $\langle p_1, p_5\rangle$ and $\langle p_3, p_4\rangle$ is tangent to any branch of $X^{st}$ at any point.
4. In this case $s_1 = s_2 \neq s_3 = s_4 = s_5$. As in case 2, we can suppose that $s_1, s_3 \in \langle s+t\rangle$. Hence, we have 4 non-infinitely near $p_i$'s (the is similar to the proof of case 1) and, for instance, $p_2$ is infinitely near of order 1 to $p_5$ which is then a tacnode or cusp of higher order of $X^{st}$. That the configuration of double points of $X^{st}$ is as asserted is now an easy consequence of the fact that this case is a degeneration of case 2.
5. In this case $s_1 = s_2 = s_3 = s_4 \neq s_5$. Supposing again $s_1 \in \langle s+t\rangle$, we have three non-infinitely near double points for $X^{st}$, say $p_1, p_2, p_5$ which are collinear (see the proof of case 1). Since this is a degeneration of case 2 in which $q_1$ and $q_3$ have come together, we have, for instance, $p_3$ is infinitely near of order 1 to $p_1$ and $p_4$ is infinitely near of order 1 to $p_2$.
6. This case is analogous to the previous case: it is a degeneration of case 4 in which $q_1$ and $q_3$ have come together.
7. This case is a degeneration of case 1 in which the quadric $q_1$ has rank 3. So $p_3, p_4, p_5$ are collinear and $p_1'+p_1'' = p_2'+p_2''$. For instance, $p_2$ is infinitely near to $p_1$ which is a then tacnode or cusp of higher order. Also, since $2(s+t+p_1'+p_1'')$ is a canonical divisor, it spans a hyperplane in $\mathbb{P}^4$ and the tangent line to the branch(es) of $X^{st}$ at $p_1$ also contains $\bar{s}$ and $\bar{t}$.

Cases $8, 10, 11, 12, 13$ are degenerations, respectively, of cases $2, 3, 4, 5, 6$ in which $q_1$ has rank 3. After exchanging $(p_1, p_2)$ with $(p_3, p_4)$ in cases 9 and 15, cases $9, 14, 15$ are degenerations of cases $4, 8, 11$ in which $q_4$ has rank 3. All these cases are similar to case 7. Q.E.D.

**Remark 2.4.** *Since the only configurations of double points (of $X^{st}$) in Lemma 2.3 that have moduli are 1 and 7, we see that, if the number of distinct points of intersection of $Q$ with $l$ is at most* 3*, then the projective equivalence class of $S(l)$ only depends on the* type *of $Q \cap l$ (see definition 2.2).*

## 3. The Chow group of $U$

We can now proceed to cut $U$ into a union of locally closed subvarieties with trivial Chow groups. The classes of the closures of these locally closed subvarieties will then generate $A_*(U)$. We will show that these classes are polynomials in the tautological classes, henceforth proving that $A_*(U)$ is generated by tautological classes. We will then use this to show that, in fact, $A_*(U) = 0$.

Below we define our strata. For each subvariety $P$ of $U$ that we define, we denote by $\overline{P}$ the closure of $P$ in $U$.

$U_0$: open subvariety of $U$ parametrizing curves whose associated quintic $Q$ is smooth, has no hyperflexes and no flex bitangents.

$U_1$: locally closed subvariety of $U$ parametrizing curves such that $Q$ is smooth, has a flex bitangent and has no hyperflex of contact 5.

$V_1$: locally closed subvariety of $U$ parametrizing curves such that $Q$ is smooth, has a hyperflex of contact 4 and has no hyperflex of contact 5.

$V'_6$: closed subvariety of $V_1$ parametrizing curves such that for some $s+t$ such that $X^{st}$ has the configuration of double points in Lemma 2.3,5, we have $p''_1 = p'_1 = u_1$, $p''_2 = p'_2 = u_2$ and $s = t$.

$V'_5$: closed subvariety of $V_1 \setminus V'_6$ parametrizing curves such that for some $s+t$ such that $X^{st}$ has the configuration of double points in Lemma 2.3,5, we have $p''_1 = p'_1 = u_1$ and $p''_2 = p'_2 = u_2$ (where $u_i + v_i \equiv K_X - s - t - 2p'_i - 2p''_i$ for $i = 1, 2$).

$V'_4$: closed subvariety of $V_1 \setminus \overline{V}'_5$ parametrizing curves such that for some $s+t$ such that $X^{st}$ has the configuration of double points in Lemma 2.3,5, we have $p''_1 = p'_1 = u_1$ and $p''_2$ or $p'_2 = u_2$.

$V'_3$: closed subvariety of $V_1 \setminus \overline{V}'_4$ parametrizing curves such that for some $s+t$ such that $X^{st}$ has the configuration of double points in Lemma 2.3,5, we have $p''_i = p'_i = u_i$ for $i = 1$ or $2$.

$V'_2$: closed subvariety of $V_1 \setminus \overline{V}'_3$ parametrizing curves such that, for some $s+t$ such that $X^{st}$ has the configuration of double points in Lemma 2.3,5, we have $p'_i = u_i$ for $i = 1$ or $2$.

$V'_1$: $= V_1 \setminus \overline{V}'_2$.

$U_2$: locally closed subvariety of $U$ parametrizing curves such that $Q$ is smooth, has a hyperflex of contact 5 and $p'_1, p''_1 \neq u_1, v_1$ for all $s+t$ such that $X^{st}$ has the configuration of double points in Lemma 2.3,6.

$U_4$: locally closed subvariety of $\overline{U}_2$ parametrizing curves such that $Q$ is smooth and $p'_1 = p''_1 = u_1$ for some $s+t$ such that $X^{st}$ has the configuration of double points in Lemma 2.3,6.

$U_3$: locally closed subvariety of $\overline{U}_2 \setminus \overline{U}_4$ parametrizing curves such that $Q$ is smooth and $p''_1 \neq p'_1 = u_1$ for some $s+t$ such that $X^{st}$ has the configuration of double points in Lemma 2.3,6.

$U_{null}$: closed subvariety of $U$ parametrizing curves with a vanishing theta-null, i.e., with $Q$ singular.

$V_3$: locally closed subvariety of $U_{null}$ parametrizing curves with two vanishing theta-nulls or more, such that (at least) one of the lines through two of the nodes of $Q$ is tangent to one of the branches of $Q$ at one of the nodes and $s \neq t$ for all $s+t$ such that $X^{st}$ has the configuration of double points in Lemma 2.3,15.

$V_5$: closed subvariety of $\overline{V}_3$ parametrizing curves such that $p_1' = p_1''$ and $s = t$ for some $s + t$ such that $X^{st}$ has the configuration of double points in Lemma 2.3,15.

$V_4$: closed subvariety of $\overline{V}_3 \setminus \overline{V}_5$ parametrizing curves such that $s = t$ for some $s + t$ such that $X^{st}$ has the configuration of double points in Lemma 2.3,15.

$Z_2$: closed subvariety of $U_{null} \setminus \overline{V}_3$ parametrizing curves with two vanishing theta-nulls or more.

$W_3$: locally closed subvariety of $U_{null}$ parametrizing curves such that the line tangent to one of the branches of $Q$ at its node has contact of order 4 with that branch and $s \neq t$ on $X$ for all $s + t$ such that $X^{st}$ has the configuration in Lemma 2.3,13.

$W_4$: closed subvariety of $\overline{W}_3$ such that $s = t$ for some $s + t$ as in the previous case.

$Y_2$: closed subvariety of $U_{null} \setminus \overline{W}_3$ parametrizing curves such that the line tangent to one of the branches of $Q$ at its node has contact of order 3 with that branch.

$W_2$: closed subvariety of $U_{null} \setminus (\overline{Z}_2 \cup \overline{W}_3)$ parametrizing curves such that the line tangent to one of the branches of $Q$ at its node is tangent to $Q$ at a smooth point.

$V_2$: closed subvariety of $U_{null} \setminus (\overline{V}_3 \cup \overline{W}_3)$ parametrizing curves such that a flex line of $Q$ (at a smooth point) passes through a node of $Q$.

$W_1$: $= U_{null} \setminus (\overline{V}_2 \cup \overline{Y}_2 \cup \overline{W}_2 \cup \overline{Z}_2)$.

Note that in the above list the subscript of any letter is the codimension of the corresponding subvariety. We will now use what we proved earlier to show:

**Theorem 3.1.** *The varieties $U_i, V_i', V_2, V_3, V_4, W_i, Y_i, Z_2$ are irreducible and have trivial Chow groups.*

*Proof:* In this preliminary part of the proof only, we will denote by $P$ any of the subvarieties in the statement of the theorem. To prove the theorem, we will produce a nonempty open subset $\widehat{P}$ of some projective space which maps onto $P$ by a finite morphism. The irreducibility of $P$ will immediately follow from that of $\widehat{P}$. The complement of $\widehat{P}$ in its projective space will have codimension 1 so we can deduce that the Chow ring with rational coefficients of $\widehat{P}$ is trivial. It then follows from Lemma A, page 332 in [9], that the Chow ring of $P$ is trivial as well.

By Remark 2.4, there is a *unique* (up to projective equivalence) quartic Del Pezzo surface $S(P)$ such that, for every $X$ such that $m_X$ (the moduli point of $X$, see "Notation and Conventions") is in $P$, the canonical model of $X$ embeds in $S(P)$. (For instance: if $P = U_0$, then $S(P)$ is the base locus of a pencil of quadrics parametrized by a line which is bitangent to $Q$ ; if $P = U_1$, then $S(P)$ is the base locus of a pencil of quadrics parametrized by a line which is a flex bitangent of $Q$.) We let $l(P)$ denote a line in $S(P)$ such that by projecting from it we obtain the corresponding configuration of points in Lemma 2.3. Fix a projective embedding of $S(P)$. Let $L(P)$ denote the pencil of quadrics containing $S(P)$ and let $\mathbb{P}(P)$ be the quotient of the linear system $|\mathcal{O}_{\mathbb{P}^4}(2)|$ of quadrics in $\mathbb{P}^4$ by $L(P)$. Then $\mathbb{P}(P)$ is a projective space of dimension 12 and its elements can be identified with nets of quadrics containing $L(P)$. Let $\widetilde{P}$ be the open subset of $\mathbb{P}(P)$ parametrizing nets of quadrics whose base locus is a smooth curve with moduli point in $P$. Then we have a canonical morphism $\widetilde{P} \longrightarrow P$ which to each net of quadrics associates (the isomorphism class of) its base locus. The variety $\widehat{P}$ (defined below and mentioned above) is the intersection of $\widetilde{P}$ with a linear subspace of $\mathbb{P}(P)$. In each case, the morphism $\widehat{P} \longrightarrow P$ is finite because it is quasi-finite and proper. The quasi-finiteness will follow from the fact that the elements of $\widehat{P}$ have finite stabilizers in the automorphism group of $S(P)$ and the properness will follow from the fact that the only time the cardinality of the fibers of $\widehat{P} \longrightarrow P$ goes down by specialization is when $\widehat{P} \longrightarrow P$ ramifies. We give a complete proof of the finiteness of the restriction morphism $\widehat{P} \longrightarrow P$ only in the case $P = U_0$ since all the other cases are analogous.

We now proceed with the proof case by case of the theorem.

$U_0$: The surface $S(U_0)$ is described in Lemma 2.3,2. The canonical morphism $\widetilde{U}_0 \longrightarrow U_0$ is finite (so $\widehat{U}_0 = \widetilde{U}_0$). i.e., quasi-finite and proper.

To show that it is quasi-finite we show that for every curve $X$ in $S(U_0)$ there are at most a finite number of curves $Y$ in $S(U_0)$ which are projectively equivalent to $X$. Let $g_Y$ be the projective transformation sending $Y$ to $X$. Then either $g_Y$ preserves $S(U_0)$ or $g_Y$ sends $S(U_0)$ to another Del Pezzo surface containing $X$. In the latter case $g_Y(S(U_0))$ is the base locus of a pencil of quadrics containing $X$ which is distinct from $L(U_0)$ and is simply bitangent to $Q$ by Lemma 2.3. Since $Q$ has a finite number of bitangents, we have only a finite number of curves $Y$ such that $g_Y$ does not preserve $S(U_0)$. There are also a finite number of curves $Y$ such that $g_Y$ does preserve $S(U_0)$ because the automorphism group of $S(U_0)$ is finite (easy to check: an automorphism of $S(U_0)$ comes from an automorphism of $\mathbb{P}^2$ which fixes $p_5$ and preserves the sets $\{p_1, p_2\}$ and $\{p_3, p_4\}$ in Lemma 2.3,2).

To show that the morphism is proper, we observe that it is separated and of finite type and use the valuative criterion of properness:

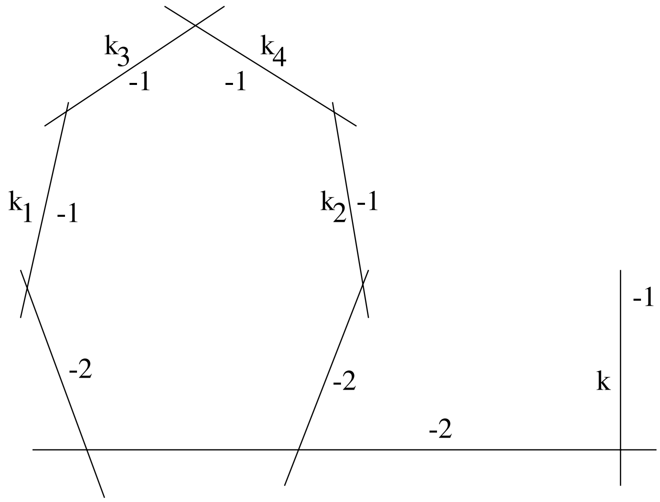
Let $\operatorname{Spec} R$ be the spectrum of a discrete valuation ring $R$ with generic point $\eta$ and closed point $r$. Let $\mathcal{X}$ be a smooth curve of genus 5 over $\operatorname{Spec} R$ such that the image of the morphism $\phi : \operatorname{Spec} R \longrightarrow \mathcal{M}_5$ obtained from $\mathcal{X}$ lies in $U_0$ and we are given a lift $\psi_\eta : \eta \longrightarrow \widetilde{U}_0$ of the restriction of $\phi$ to $\eta$. We need to extend $\psi_\eta$ to all of $\operatorname{Spec} R$. The data of $\psi_\eta$ is equivalent to the data of a birational map $f_\eta : \mathcal{X}_\eta \longrightarrow \mathbb{P}^2_\eta$ such that the configuration of double points of the image of $\mathcal{X}_\eta$ is as in Lemma 2.3,2. This map specializes to a map $f_r : \mathcal{X}_r \longrightarrow \mathbb{P}^2_r$ which is birational because $\mathcal{X}_r$ is not trigonal, nor bielliptic nor hyperelliptic. The configuration of double points of the image of $\mathcal{X}_r$ by $f_r$ is a limit of the configuration of double points of $f_\eta(\mathcal{X}_\eta)$ and it cannot be more degenerate than this latter because the image of $\phi$ is contained in $U_0$. Thus the configuration of double points of $f_r(\mathcal{X}_r)$ is the same as that of $f_\eta(\mathcal{X}_\eta)$. Thus we have extended $\psi_\eta$ to $\operatorname{Spec} R$.

$U_1$: The surface $S(U_1)$ is described in Lemma 2.3,4. The fibers of the morphism $\widetilde{U}_1 \longrightarrow U_1$ have dimension at least one. Since $Q$ has a finite number of flex bitangents, the positive-dimensional fibers come from the automorphism group of $S(U_1)$. Hence, since the general fibers are one-dimensional, all the fibers are one-dimensional. The automorphism group of $S(U_1)$ fixes the double points of $S(U_1)$ since the double points are of different kinds. So the automorphism group of $S(U_1)$ preserves $l(U_1)$ (since $l(U_1)$ contains the two double points, see Lemma 2.3,4).

We claim that the only fixed points on $l(U_1)$ of any one-dimensional subgroup of the automorphism group of $S(U_1)$ are the two double points: If such a group has a third fixed point on $l(U_1)$, then it must fix every point on $l(U_1)$. The automorphism group is generated by an element $f$ of $PGL_3$ which fixes $p_1$ and $p_5$ and exchanges $p_3$ and $p_4$ and the subgroup $G$ of $PGL_3$ which fixes the points $p_1, p_5, p_3, p_4$ or, equivalently, the line $\langle p_3, p_4 \rangle$ and the point $p_1$. Therefore $G$ is the only one-dimensional proper subgroup of the automorphism group. A conic $C$ in $\mathbb{P}^2$ which is tangent to $\langle p_1, p_5 \rangle$ at $p_5$, contains $p_1$ and $p_3$ and is general for this property is not globally invariant under $G$ since its second point of intersection with $\langle p_1, p_4 \rangle$ is not fixed by $G$. The conic $C$ and its image by a general element of $G$ have order of contact 2 at $p_5$, hence their strict transforms in $S(U_1)$ intersect $l(U_1)$ in two distinct points which are then not fixed under $G$ (or the full automorphism group).

Fix a point $p \in l(U_1)$ distinct from the double points of $S(U_1)$. Since the stabilizer of $p$ in the automorphism group of $S(U_1)$ is finite, for each curve $X$ with $m_X \in U_1$, only a finite number of curves in $S(U_1)$ and isomorphic to $X$ contain $p$. We let $\widehat{U}_1$ be the subvariety of $\widetilde{U}_1$ parametrizing nets of quadrics through $p$ (note that none of the curves $X$ with $m_X \in U_1$ contains any double point of $S(U_1)$ because otherwise $X$ would not be smooth).

$V_i'(1 \le i \le 6)$: The surface $S(V_1) = S(V_i')$ is described in Lemma 2.3,5. The configuration of curves with negative self-intersection on the minimal desingularization of $S(V_1)$ is the following




With our previous notation (in Lemma 2.3 and the definition of $V_1'$), the line $k_1$ (resp. $k_2$) above is the strict transform of $\langle p_1' + p_1'' \rangle$ (resp. $\langle p_2' + p_2'' \rangle$), the line $k_3$ (resp. $k_4$) is the strict transform of $\langle u_1 + v_1 \rangle$ (resp. $\langle u_2 + v_2 \rangle$) and $k = l(V_1) = l(V_i')$ is the strict transform of $\langle s + t \rangle$.

Let $m_1$ and $m_2$ be the lines (in $\mathbb{P}^2$) through $p_1$ and $p_2$ with tangent directions given by $p_3$ and $p_4$ respectively. The data of an automorphism of $S(V_1)$ is equivalent to the data of an automorphism of $\mathbb{P}^2$ which fixes $p_5$ and preserves or exchanges the pairs $(p_1, m_1)$ and $(p_2, m_2)$. Equivalently, the automorphism of $\mathbb{P}^2$ must fix $p_5$ and the point $p_0 := m_1 \cap m_3$ and must either fix or exchange $p_1$ and $p_2$. It follows that any automorphism of $S(V_1)$ lifts to an automorphism of its minimal desingularization which leaves $k$ invariant, fixes the point $k_3 \cap k_4$ (inverse image of $p_0$) and leaves invariant or exchanges $(k_1, k_3)$ and $(k_2, k_4)$. In particular, there is a one-dimensional subgroup (of index two) of the automorphism group of $S(V_1)$ which fixes $k_1 \cap k_3$ and $k_2 \cap k_4$. It can be seen as in the case of $U_1$ that the point of intersection with the $-2$ curve and $k_1 \cap k_3$ (resp. $k_2 \cap k_4$) are the only points on $k_1$ (resp. $k_2$) which are fixed under a one-dimensional subgroup of the automorphism group of $S(V_1)$. None of our smooth curves of genus 5 passes through the double point of $S(V_1)$ so their strict transforms do not intersect the $(-2)$-curves in the picture.

- $V_1'$: Choose a general point $p$ on $k_1$ and let $\widehat{V}_1'$ be the subvariety of $\widetilde{V}_1$ parametrizing nets of quadrics through (the image of) $p$ (in $S(V_1)$).
- $V_2'$: Let $\widehat{V}_2'$ be the subvariety of $\widetilde{V}_1$ parametrizing nets of quadrics through (the images of) $p$ and $k_1 \cap k_3$ (in $S(V_1)$).
- $V_3'$: Choose a general point $q$ on $k_2$. Let $\widehat{V}_3'$ be the subvariety of $\widetilde{V}_1$ parametrizing nets of quadrics through $q$ and tangent to $k_1$ at $k_1 \cap k_3$.
- $V_4'$: Let $\widehat{V}_4'$ be the subvariety of $\widetilde{V}_1$ parametrizing nets of quadrics through $q$ and $k_2 \cap k_4$ and tangent to $k_1$ at $k_1 \cap k_3$.
- $V_5'$: Choose a general conic $C$ which corresponds to a line in $\mathbb{P}^2$ through $p_1$. Let $\widehat{V}_5'$ be the subvariety of $\widetilde{V}_1$ parametrizing nets of quadrics tangent to $C$ and tangent to $k_i$ at $k_i \cap k_{i+2}$ for $i = 1$ and $i = 2$.
- $V_6'$: Let $\widehat{V}_6'$ be the subvariety of $\widetilde{V}_1$ parametrizing nets of quadrics tangent to $C$, tangent to $k$ and tangent to $k_i$ at $k_i \cap k_{i+2}$ for $i = 1, 2$.

$W_1$: The surface $S(W_1)$ is described in Lemma 2.3,8: it has three nodes and contains two of the lines through the three nodes. By our choices, $l(W_1)$ is one of these two lines. Choose a point

$p$ on $l(W_1)$ distinct from the nodes. As in the case of $U_1$, we can let $\widehat{W}_1$ be the subvariety of $\widetilde{W}_1$ parametrizing nets of quadrics through $p$.

$U_2$: The surface $S(U_2)$ is described in Lemma 2.3,6: it has one double point, contains 3 lines one of which ($l(U_2)$) intersects only one other line (at the double point), say $l'(= \langle p'_1 + p''_1 \rangle)$. The line $l'$ intersects the third line, say $l''$, outside the double point. Fix two general points $p$ and $q$ on $l(U_2)$ and $l'$ respectively and let $\widehat{U}_2$ be the subvariety of $\widetilde{U}_2$ parametrizing nets of quadrics through $p$ and $q$.

$U_3$: We have $S(U_3) = S(U_2)$. Let $\widehat{U}_2$ be the subvariety of $\widetilde{U}_3$ parametrizing nets of quadrics through $p$, $q$ and $l' \cap l''$.

$U_4$: We have $S(U_4) = S(U_2)$. Choose a general point $r$ on $l''$. Let $\widehat{U}_4$ be the subvariety of $\widetilde{U}_2$ parametrizing nets of quadrics through $q$ and $r$ and tangent to $l'$ at $l' \cap l''$.

$V_2$: The surface $S(V_2)$ is described in Lemma 2.3,9: the two lines $\langle o_i, Sing(q_3) \rangle$ (see 2.3,9) do not intersect any lines in $S(V_2)$ (other than themselves) outside the double points of $S(V_2)$ and each contain two double points of $S(V_2)$. Choose general points $p$ and $q$, one on each line. Let $\widehat{V}_2$ be the subvariety of $\widetilde{V}_2$ parametrizing nets of quadrics through $p$ and $q$.

$W_2, Y_2, Z_2$: The surfaces $S(W_2), S(Y_2), S(Z_2)$ are described in Lemma 2.3 respectively in cases $11, 12, 14$. Each surface contains (at least) two lines which do not contain any point (other than a double point) fixed under a two-dimensional subgroup of the automorphism group of the surface. These cases are analogous to the case of $V_2$.

$W_3$: The surface $S(W_3)$ is described in Lemma 2.3,13. It contains one line ($l(W_3)$) and one pencil of conics. The irreducible conics in this pencil intersect $l(W_3)$ only at the double point of $S(W_3)$. Choose an irreducible conic $C$ on $S(W_3)$ and a point $r$ on $C$ distinct from the double point of $S(W_4)$. Choose distinct points $p$ and $q$ (distinct from the double point) on $l(W_3)$. Let $\widehat{W}_3$ be the subvariety of $\widetilde{W}_3$ parametrizing nets of quadrics through $p$, $q$ and $r$.

$W_4$: We have $S(W_4) = S(W_3)$. Let $\widehat{W}_4$ be the subvariety of $\widetilde{W}_4$ parametrizing nets of quadrics tangent to $l(W_3)$ at $p$ and tangent to $C$ at $r$.

$V_3$: The surface $S(V_3)$ is described in Lemma 2.3,15. It contains two lines which do not contain any point (other than a double point) fixed under a three-dimensional subgroup of the automorphism group of the surface. Choose two general points on the line $l(V_3)$ (equal to $\langle s + t \rangle$) and a general point on the second line $l'$. Let $\widehat{V}_3$ be the subvariety of $\widetilde{V}_3$ parametrizing nets of quadrics through these three points.

$V_4$: We have $S(V_3) = S(V_2)$. Choose a general point $p$ on $l(V_3)$, two general point $q$ and $r$ on the second line $l'$. Let $\widehat{V}_4$ be the subvariety of $\widetilde{V}_3$ parametrizing nets of quadrics through $q$ and $r$ and tangent to $l(V_3)$ at $p$.

$V_5$: The surface $S(V_5) = S(V_2)$ has two pencils of conics whose smooth elements intersect the two lines in $S(V_3)$ only at the double points. Choose a general conic $C$ from one of these pencils. Let $\widehat{V}_5$ be the subvariety of $\widetilde{V}_3$ parametrizing nets of quadrics tangent to $C$, tangent to $l(V_3)$ at $p$ and tangent to $l'$ at $q$. Q.E.D.

It follows from the above theorem that $[\overline{U}_i]_U, [\overline{V}_i]_U, [\overline{V}'_i]_U, [\overline{W}_i]_U, [\overline{Y}_i]_U, [\overline{Z}_2]_U$ generate $A_*(U)$. In [17] (page 299) Mumford defined a set of tautological classes in the Chow ring of $\overline{\mathcal{M}}_g$. We recall their definition:

Let $\overline{\mathcal{C}}_g$ be the coarse moduli space of one-pointed stable curves of genus $g$. Let $\omega$ be the relative dualizing sheaf of the natural morphism $\pi : \overline{\mathcal{C}}_g \longrightarrow \overline{\mathcal{M}}_g$ and put $K = c_1(\omega)$. Then

$$\kappa_l = \pi_* K^{l+1}, \ \lambda = c_1(\pi_*\omega), \ \lambda_l = c_l(\pi_*\omega) \ for \ l > 1.$$

Conjecturally (see [17] page 272), the restrictions of these classes to $\mathcal{M}_g$ generate the stable cohomology of $\mathcal{M}_g$, i.e., when $g$ is big compared to $k$, then $H^{2k}(\mathcal{M}_g)$ is generated by these classes and $H^{2k+1}(\mathcal{M}_g) = 0$.

We denote by the same symbols the restrictions of the above classes to any subvariety of $\mathcal{M}_g$ (for $g = 3$ or 5). For a subvariety $A$ of $\overline{\mathcal{M}}_g$, we denote by $\mathcal{C}_A$ the restriction of $\overline{\mathcal{C}}_g$ to $A$ and by $\mathcal{C}_A^n$ its $n$-th fiber product over $A$. For each subset $I = \{i_1, \ldots, i_r\}$ of $\{1, \ldots, n\}$ we let $\Delta_I = \Delta_{i_1,\ldots,i_r}$ be the diagonal of $\mathcal{C}_A^n$ parametrizing elements $(x_1, \ldots, x_n)$ in the fibers of $\mathcal{C}_A^n \longrightarrow A$ such that $x_{i_1} = \ldots = x_{i_r}$. We also let $K_i$ be the first Chern class of the pull-back of $\omega$ by the $i$-th projection $\mathcal{C}_A^n \longrightarrow \mathcal{C}_A$. We now show

**Theorem 3.2.** *The classes $[\overline{U}_i]_U, [\overline{V}_i]_U, [\overline{V}'_i]_U, [\overline{W}_i]_U, [\overline{Y}_i]_U, [\overline{Z}_2]_U$ are polynomials in the tautological classes.*

*Proof:* We will characterize the points of each variety in terms of the existence of points (or divisors) with certain properties on the corresponding curves. Then we will use determinantal formulas to show that the classes are combinations of tautological classes. In order to be able to apply the determinantal formulas, we need $U$ to be Cohen-Macaulay. This is certainly the case in characteristic zero but we do not know whether the same holds in positive characteristic (since the quotient of a smooth variety by a finite group may not be Cohen-Macaulay if the characteristic of the base field $k$ divides the order of the group). Therefore we will need to replace $U$ by a smooth finite cover and all the strata by their inverse images in this finite cover. By Lemma A page 332 in [9], if we show that the classes of the inverse images of the (closures of the) strata are combinations of the (inverse images of the) tautological classes, then it will follow that the classes of the (closures of the) strata themselves are combinations of the tautological classes. To avoid notational complications we will only use this finite cover implicitly, i.e., denote the finite cover of $U$ and the inverse images of the strata by the same letters. This finite cover will only be used when we are applying determinantal formulas in Chern classes of bundles.

Consider the configuration of the double points of $X^{st}$ given in Lemma 2.3,2. The configuration of points for any of our strata is a limit of this configuration in which some of the points have come together. We will define a finite cover $\mathcal{P}$ of $U$ which, roughly speaking, parametrizes the points in configuration 2.3,2, sits inside $\mathcal{C}_U^{14}$ and whose intersections with various diagonals will map onto the closures of our strata.

Consider the codimension 14 subvariety of $\mathcal{C}_U^{14}$ parametrizing 14-tuples $(s, t, p'_1, p''_1, \ldots, p'_4, p''_4, u_1, v_1, u_2, v_2)$ of points on curves $X$ ($m_X \in U$) such that

$$\left\{\begin{array}{l} h^0(2s+2t) \geq 2 \ , \ h^0(s+t+p'_1+p''_1) \geq 2 \ , \\ h^0(s+t+p'_1+p''_1+p'_2+p''_2) \geq 3 \ , \\ h^0(2s+t+p'_1+p''_1+p'_2+p''_2) \geq 4 \ , \\ h^0(s+t+p'_1+p'_3+u_1+v_1) \geq 3 \ , \\ h^0(s+t+p'_3+p''_3) \geq 2 \ , \\ h^0(s+t+p'_3+p''_3+p'_4+p''_4) \geq 3 \ , \\ h^0(2s+t+p'_3+p''_3+p'_4+p''_4) \geq 4 \ , \\ h^0(s+t+p'_2+p'_4+u_2+v_2) \geq 3. \end{array}\right. \tag{1}$$

The condition $h^0(2s+2t) \geq 2$ insures that $\overline{s} = \overline{t}$ so that we have the configuration of double points in Lemma 2.3,2 (or a degeneration of it) for the double points of $X^{st}$. The condition $h^0(s+t+p'_1+p''_1) \geq 2$ means that $p'_1$ and $p''_1$ map to a double point $p_1$ of $X^{st}$ and the condition $h^0(s+t+p'_1+p''_1+p'_2+p''_2) \geq 3$ means that $p_1$ and the images of $p'_2$ and $p''_2$ in $|K_X - s - t|^*$ are collinear. Then $h^0(2s+t+p'_1+p''_1+p'_2+p''_2) \geq 4$ means that $\overline{s}, p_1$ and the images of $p'_2$ and $p''_2$ are collinear which implies that $p_2'$ and $p_2''$ map to the double point $p_2$ of $X^{st}$ since the line $\langle \overline{s}, p_1\rangle$ intersects $X^{st}$ in $\overline{s}$, $p_1$ and $p_2$. Finally, the condition $h^0(s+t+p_1'+p_3'+u_1+v_1)\geq 2$ means that $p_1,p_3$ and the images of $u$ and $v$ are collinear.

The locus of divisors of the form $\sum_{i=1}^r n_ir_i$ with $0\leq n_i\in\mathbb{Z}$ and $r_i\in X$, $m_X\in U$, for all $i$, such that $\sum_{i=1}^r n_i = n$ and $h^0(\sum_{i=1}^r n_ir_i)\geq k$ is a degeneracy locus for a map of vector bundles in the following way: Consider the exact sequence

$$0\longrightarrow \mathcal{O}_{\mathcal{C}_U^{r+1}}\longrightarrow \mathcal{O}_{\mathcal{C}_U^{r+1}}\left(\sum_{i=1}^r n_i\Delta_{i,r+1}\right)\longrightarrow \frac{\mathcal{O}_{\mathcal{C}_U^{r+1}}\left(\sum_{i=1}^r n_i\Delta_{i,r+1}\right)}{\mathcal{O}_{\mathcal{C}_U^{r+1}}}\longrightarrow 0.$$

The sequence of higher direct images by the projection $\rho$ to the first $r$ factors is

$$0\longrightarrow \mathcal{O}_{\mathcal{C}_U^{r}}\longrightarrow \rho_*\left(\mathcal{O}_{\mathcal{C}_U^{r+1}}\left(\sum_{i=1}^r n_i\Delta_{i,r+1}\right)\right)\longrightarrow \rho_*\left(\frac{\mathcal{O}_{\mathcal{C}_U^{r+1}}\left(\sum_{i=1}^r n_i\Delta_{i,r+1}\right)}{\mathcal{O}_{\mathcal{C}_U^{r+1}}}\right)$$

$$\longrightarrow R^1\rho_*\mathcal{O}_{\mathcal{C}_U^{r+1}}\longrightarrow R^1\rho_*\left(\mathcal{O}_{\mathcal{C}_U^{r+1}}\left(\sum_{i=1}^r n_i\Delta_{i,r+1}\right)\right)\longrightarrow 0.$$

So we see that the locus where $h^0(\sum_{i=1}^r n_ir_i)\geq k$ is the locus where the rank of the map $\rho_*\left(\mathcal{O}_{\mathcal{C}_U^{r+1}}\left(\sum_{i=1}^r n_i\Delta_{i,r+1}\right)\right.$ $R^1\rho_*\mathcal{O}_{\mathcal{C}_U^{r+1}}$ is less than or equal to $n-k+1$.

Therefore our subvariety above is the degeneracy locus for the following flags and maps of vector bundles

$$\begin{array}{rcl}
A:=\rho_*\left(\frac{\mathcal{O}_{\mathcal{C}_U^{15}}(2\Delta_{1,15}+2\Delta_{2,15})}{\mathcal{O}_{\mathcal{C}_U^{15}}}\right) & \longrightarrow & F:=R^1\rho_*\mathcal{O}_{\mathcal{C}_U^{15}}\,,\\
A_1^i:=\rho_*\left(\frac{\mathcal{O}_{\mathcal{C}_U^{15}}(\Delta_{1,15}+\Delta_{2,15}+\Delta_{i+3,15}+\Delta_{i+4,15})}{\mathcal{O}_{\mathcal{C}_U^{15}}}\right) & \subset & \\
\subset\ A_2^i:=\rho_*\left(\frac{\mathcal{O}_{\mathcal{C}_U^{15}}(\Delta_{1,15}+\Delta_{2,15}+\Delta_{i+3,15}+\Delta_{i+4,15}+\Delta_{i+5,15}+\Delta_{i+6,15})}{\mathcal{O}_{\mathcal{C}_U^{15}}}\right) & \subset & \\
\subset\ A_3^i:=\rho_*\left(\frac{\mathcal{O}_{\mathcal{C}_U^{15}}(2\Delta_{1,15}+\Delta_{2,15}+\Delta_{i+3,15}+\Delta_{i+4,15}+\Delta_{i+5,15}+\Delta_{i+6,15})}{\mathcal{O}_{\mathcal{C}_U^{15}}}\right) & \longrightarrow & R^1\rho_*\mathcal{O}_{\mathcal{C}_U^{15}}\,,\\
E^j:=\rho_*\left(\frac{\mathcal{O}_{\mathcal{C}_U^{15}}(\Delta_{1,15}+\Delta_{2,15}\Delta_{3+j,15}+\Delta_{7+j,15}+\Delta_{11+j,15}+\Delta_{12+j,15})}{\mathcal{O}_{\mathcal{C}_U^{15}}}\right) & \longrightarrow & R^1\rho_*\mathcal{O}_{\mathcal{C}_U^{15}}\,,
\end{array}$$

for $i=0,4$ and $j=0,2$, where the ranks of $A\longrightarrow F$ and $A_1^i\longrightarrow F$ are at most 3 and the ranks of $A_2^i\longrightarrow F$, $A_3^i\longrightarrow F$ and $E^j\longrightarrow F$ are at most 4. By [14] (proof of Lemma 2.9), for $X$ non-bielliptic, there are only a finite number of divisors $s+t$ such that $h^0(2s+2t)=2$. For fixed $s+t$, there are only a finite number of divisors $p_i'+p_i''$ with $i=1,2,3,4$ with the properties (1) since, as we saw, then $p_i:=\pi_{st}(p_i')=\pi_{st}(p_i'')$ would be a double point of $X^{st}$ for $i=1,2,3,4$. Hence this locus maps onto $U$ with finite fibers everywhere. So this locus has the right codimension as given in [10] pages 249-250 and 254. Therefore we can use the formulae on pages 249-250 and 254 of [10] to compute the class of this locus as a (Schur) polynomial in the Chern classes of the vector bundles $A, A_k^i, E^j$ and $F$. Now, in the same way as in [17] pages 310-314, we see that the Chern classes of these vector bundles are polynomials in the classes of the diagonals and the first Chern classes $K_i$, $1\leq i\leq 14$, of the pull-backs of $\omega$ by the projections $\mathcal{C}_U^{14}\longrightarrow\mathcal{C}_U$, for instance:

$$c(A)=(1-K_1)(1-2K_1)(1-K_2+2\Delta_{1,2})(1-2K_2+2\Delta_{1,2}).$$

However, the above locus contains the codimension 14 loci of 14-tuples of the form, for instance, $(s,t,p_1',p_1'',p_2',p_2'',p_3',p_3'',p_4',p_4'',p_1'',p_3'',p_2'',p_4'')$ with the same conditions as above on $s,t$ and $p_i',p_i''$. These loci are of no interest to us. The class of the latter can be expressed as the product of $[\Delta_{4,11}\cap$

$\Delta_{8,12} \cap \Delta_{6,13} \cap \Delta_{10,14}]_{\mathcal{C}^{12}}$ by the class of the locus parametrizing 14-tuples $(s, t, p'_1, p''_1, p'_2, p''_2, p'_3, p''_3, p'_4, p''_4, u_1, v_1, u_2, v_2)$ with

$$h^0(2s+2t) \geq 2 \ , \ h^0(s+t+p'_i+p''_i) \geq 2 \ ,$$
$$h^0(s+t+p'_i+p''_i+p'_{i+1}+p''_{i+1}) \geq 3$$
$$\text{and} \quad h^0(2s+t+p'_i+p''_i+p'_{i+1}+p''_{i+1}) \geq 4 \, ,$$

for $i = 1, 3$. So the classes of the unwanted loci are also expressible in terms of the classes of the diagonals and the $K_i$'s. Therefore the class of the closure $\mathcal{P}$ of the subvariety parametrizing 14-tuples of *distinct* points with property (1) above is also a polynomial in the classes of the diagonals and the $K_i$'s.

It is seen now, in the same way as in [13] page 55 that the class of the pushforward to $U$ of the intersection of some of the diagonals with $\mathcal{P}''$ is a polynomial in the tautological classes. So all we need to do is to express the closures of all of our strata as such a push-forward.

$\overline{U}_1$: The moduli point of $X$ is in $U_1$ when, for some divisor $s+t$ on $X$, the curve $X^{st}$ has the configuration of points in Lemma 2.3,4. This is the limit of configuration 2 (in Lemma 2.3) where $\overline{s}$ and $p_2$ have come together. Therefore $\overline{U}_1$ is the push-forward to $U$ of $\Delta_{1,5} \cap \mathcal{P}''$.

$\overline{V}_1$: The moduli point of $X$ is in $V_1$ when, for some divisor $s+t$ on $X$, the curve $X^{st}$ has the configuration of points in Lemma 2.3,5. This is the limit of configuration 2.3,2 in which $p_1$ and $p_3$ have come together as well as $p_2$ and $p_4$. We claim that the intersection $\Delta_{3,7} \cap \mathcal{P}$ pushes forward to $\overline{V}_1$. Indeed, the only specializations of configuration 2 in Lemma 2.3 in which $p_1$ (or $p_2$) and $p_3$ (or $p_4$) have come together are configurations 5, 6, 12 and 13: these are all specializations of configuration 5 so, in all cases, the moduli point of $X$ is in $\overline{V}_1$.

$\overline{V}'_2$: The intersection $\Delta_{3,7,11} \cap \mathcal{P}$ pushes forward to $\overline{V}'_2$.

$\overline{V}'_3$: The intersection $\Delta_{3,4,7,11} \cap \mathcal{P}$ pushes forward to $\overline{V}'_3$.

$\overline{V}'_4$: The intersection $\Delta_{5,13} \cap \Delta_{3,4,7,11} \cap \mathcal{P}$ pushes forward to $\overline{V}'_4$.

$\overline{V}'_5$: The intersection $\Delta_{5,6,13} \cap \Delta_{3,4,7,11} \cap \mathcal{P}$ pushes forward to $\overline{V}'_5$.

$\overline{V}'_6$: The intersection $\Delta_{1,2} \cap \Delta_{5,6,13} \cap \Delta_{3,4,7,11} \cap \mathcal{P}$ pushes forward to $\overline{V}'_6$.

$U_{null}$: This is analogous to the case of $\overline{V}_1$ with configuration 2.3,5 replaced by 2.3,8. The intersection $\Delta_{3,5} \cap \mathcal{P}$ pushes forward to $U_{null}$.

$\overline{U}_2$: The moduli point of $X$ is in $U_2 \cup U_3 \cup U_4$ if and only if $X^{st}$ has the configuration of double points in Lemma 2.3,6. The intersection $\Delta_{1,5,9} \cap \mathcal{P}$ pushes forward to $\overline{U}_2$.

$\overline{U}_3$: The intersection $\Delta_{3,11} \cap \Delta_{1,5,9} \cap \mathcal{P}$ pushes forward to $\overline{U}_3$.

$\overline{U}_4$: The intersection $\Delta_{3,4,11} \cap \Delta_{1,5,9} \cap \mathcal{P}$ pushes forward to $\overline{U}_4$.

$\overline{V}_2$: The moduli point of $X$ is in $V_2$ if and only if $X^{st}$ has the configuration of double points in Lemma 2.3,9. The intersection $\Delta_{1,9} \cap \Delta_{3,5} \cap \mathcal{P}$ pushes forward to $\overline{U}_2$.

$\overline{V}_3$: The moduli point of $X$ is in $V_3 \cup V_4 \cup V_5$ if and only if $X^{st}$ has the configuration of double points in Lemma 2.3,15. The intersection $\Delta_{1,7,9} \cap \Delta_{3,5} \cap \mathcal{P}$ pushes forward to $\overline{V}_3$.

$\overline{V}_4$: The intersection $\Delta_{1,2,7,9} \cap \Delta_{3,5} \cap \mathcal{P}$ pushes forward to $\overline{V}_4$.

$\overline{V}_5$: The intersection $\Delta_{1,2,7,9} \cap \Delta_{3,4,5} \cap \mathcal{P}$ pushes forward to $\overline{V}_4$.

$\overline{W}_2$: The moduli point of $X$ is in $W_2$ if and only if $X^{st}$ has the configuration of double points in Lemma 2.3,11. The intersection $\Delta_{1,3,5} \cap \mathcal{P}$ pushes forward to $\overline{W}_2$.

$\overline{W}_3$: In this case all the points $\overline{s}, \overline{t}$ and $p_1, \dots, p_5$ coincide. The moduli point of $X$ is in $W_3 \cup W_4$ if and only if $X^{st}$ has the configuration of double points in Lemma 2.3,13. The intersection $\Delta_{1,3,5,7} \cap \mathcal{P}$ pushes forward to $\overline{W}_3$.

$\overline{W}_4$: The moduli point of $X$ is in $W_4$ if and only if $X^{st}$ has the configuration of double points in Lemma 2.3,13 and $s = t$. The intersection $\Delta_{1,2,3,5,7} \cap \mathcal{P}$ pushes forward to $\overline{W}_4$.

$\overline{Y}_2$: The moduli point of $X$ is in $Y_2$ if and only if $X^{st}$ has the configuration of double points in Lemma 2.3,12. The intersection $\Delta_{3,5,7} \cap \mathcal{P}$ pushes forward to $\overline{Y}_2$.

$\overline{Z}_2$: This is the locus of curves with at least two vanishing theta-nulls. The moduli point of $X$ is in $Z_2$ if and only if $X^{st}$ has the configuration of double points in Lemma 2.3,14. The intersection $\Delta_{3,5} \cap \Delta_{7,9} \cap \mathcal{P}$ pushes forward to $\overline{V}_2$.

Q.E.D.

**Corollary 3.3.** $A_*(U) = 0$.

*Proof:* By Theorems 3.1 and 3.2, the Chow ring of $U$ is generated by tautological classes. So, by [17] page 309 and [9] page 447, the ring of tautological classes is generated by $\lambda = 12\kappa_1$ and $\kappa_2$ (by continuity, the relations given by the two authors remain valid in positive characteristic). Since the class of the locus of trigonal curves is a positive multiple of $\lambda$ by [13] page 55, we have $\lambda = \kappa_1 = 0$ in $A_*(U)$ (the proof is valid in characteristic $\neq 2$). The class of the locus of curves with a point $p$ such that $h^0(3p) \geq 2$ can be computed from formula (7.7) in [17] to be $81\kappa_2 - 25\kappa_1\lambda + 48\lambda_2$. Since this locus is contained in the trigonal locus, we have $81\kappa_2 - 25\kappa_1\lambda + 48\lambda_2 = 0$ in $A_*(U)$. Since $\lambda_2 = \lambda^2/2$, we also have $\lambda_2 = 0$. Therefore $\kappa_2 = 0$. Q.E.D.

## 4. The Chow groups of $B$ and $V$

We first show

**Theorem 4.1.** *The class of $B$ in $V := U \cup B = \mathcal{M}_5 \setminus (T \cup H)$ is a combination of tautological classes.*

*Proof:* As in the proof of Theorem 3.2, we implicitly replace $V$ by a smooth finite cover and we replace the subvarieties of $V$ by their inverse images in the smooth cover.

To show that $[B]_V$ is a combination of tautological classes we construct, as in the proof of Theorem 3.2, the subvariety $\mathcal{P}'$ of $\mathcal{C}_V^7$ as the closure of the subvariety parametrizing 7-tuples $(s, p_1', p_1'', p_2', p_2'', p_3', p_3'')$ of *distinct* points on curves $X$ (with $m_X \in V$) such that $h^0(4s) \geq 2$ and $h^0(2s + p_i' + p_i'') \geq 2$ for $i = 1, 2, 3$. As before, the class of $\mathcal{P}'$ can be expressed as a combination of the classes of the diagonals and the $K_i$'s.

The image in $V$ of the intersection $\mathcal{P}'' := \mathcal{P}' \cap \Delta_{2,3} \cap \Delta_{4,5} \cap \Delta_{6,7}$ contains $B$: Let $X \longrightarrow E$ be a bielliptic structure on a curve $X$ of genus 5 and let $s_1, \ldots, s_8$ be its ramification points. Then, for all $i, j$ between 1 and 8, we have $h^0(2s_i + 2s_j) = 2$ since $2s_i + 2s_j$ is the pullback of a divisor on $E$.

The intersection $\mathcal{P}''$ is proper:

- The map $\mathcal{P}'' \longrightarrow V$ is quasi-finite on its image: By [14] proof of Lemma 2.9, a curve $X$ of genus 5 with moduli point in $V$ has a finite number of divisors $s + t$ such that $h^0(2s + 2t) = 2$ unless it is bielliptic and then all such divisors which move in a one-dimensional family are of the form $s + \iota s$ where $\iota$ is a bielliptic involution on $X$. So to obtain $h^0(4s) \geq 2$ we must have $s = \iota s$. So $X$ has a finite number of points $s$ with $h^0(4s) = 2$ and a finite number of points $t$ with $h^0(2s + 2t) \geq 2$.
- The image $P'$ of $\mathcal{P}'' \cap C_U^7$ in $U$ has dimension 8: the variety $P'$ parametrizes curves which admit an embedding in $\mathbb{P}^2$ with configuration of double points as in Lemma 2.3,2 such that $\overline{s} = \overline{t}$ is a cusp as well as three other $p_i$'s. Therefore $P'$ has codimension 4 or dimension 8.
- $B$ has dimension 8

Therefore, the class of $\mathcal{P}''$ is a combination of the $K_i$'s and the diagonals and the class of the image of $\mathcal{P}''$ in $V$ is a combination of tautological classes. Therefore, since, by the above, $B$ is a component of the image of $\mathcal{P}''$ in $V$, we obtain that a linear combination with positive coefficients of $[B]_V$ and $[P']_V$ is a combination of tautological classes. By Corollary 3.3, the only possibly nonzero codimension 4 class in $A_*(V)$ is $[B]_V$. We deduce that $[P']_V$ is a nonnegative multiple of $[B]_V$. Therefore $[B]_V$ is a combination of tautological classes. Q.E.D.

We now proceed to find generators for $A_*(B)$. Let $B_i$ be the subvariety of $B$ parametrizing curves $X$ with $i$ distinct bielliptic structures, i.e., $Q$ contains $i$ lines. We have $B_1 = B$, $B_4 = B_5$, $\dim(B_2) = 5$, $\dim(B_3) = 3$ and $\dim(B_5) = 2$. Put $B_i^0 = B_i \setminus B_{i+1}$. For $X$ such that $m_X \in B_1^0$, let $l$ be the line in $\Pi$ which is contained in $Q$. Let $u_1, \dots, u_4$ be the nodes of $Q$ that are on the intersection of $l$ with the quartic component $R$ of $Q$ and, for each $i \in \{1, \dots, 4\}$, let $m_i$ be the line through $u_i$ which is tangent to the branch of $Q$ which belongs to $R$. We define the following strata in $B_1^0$.

**$B_{1,0}^0$:** The open subvariety of $B_1^0$ parametrizing curves such that, for all $i$, the line $m_i$ has contact of order 3 with $Q$ at $u_i$ and is elsewhere transverse to $Q$.

**$B_{1,1}^0$:** The locally closed subvariety of $B_1^0$ parametrizing curves such that for some $i$, the line $m_i$ has contact of order 4 with $Q$ at $u_i$ and for all $j$ the order of contact of $m_j$ with $Q$ at $u_j$ is at most 4.

**${B_{1,1}^0}'$:** The locally closed subvariety of $B_1^0$ parametrizing curves such that for some $i$, the line $m_i$ is tangent to $Q$ at a smooth point and, for all $j$, the order of contact of $m_j$ with $Q$ at $u_j$ is at most 4 and $m_j$ does not contain another node of $Q$.

**$B_{1,2}^0$:** The closed subvariety of $B_1^0$ parametrizing curves such that for some $i$, the line $m_i$ has contact of order 5 with $Q$ at $u_i$.

**${B_{1,2}^0}'$:** The closed subvariety of $B_1^0$ parametrizing curves such that for some $i$, the line $m_i$ contains another node of $Q$.

For $X$ such that $m_X \in B_2^0$ let $l_1$ and $l_2$ be the two lines in $Q$, let $C$ be the cubic in $Q$ and let $u_1, u_2, u_3$ (resp. $u_4, u_5, u_6$) be the points of intersection of $C$ with $l_1$ (resp. $l_2$). For $m_X$ general in $B_2^0$, there are four lines $m_{ij}$, $1 \leq j \leq 4$ through $u_i$ $(1 \leq i \leq 6)$ which are tangent to $C$ at another point. For $m_X$ not necessarily general in $B_2^0$, some of these lines may come together or their point of tangency may coincide with $u_i$ which is then a flex of $C$. We define the following strata in $B_2^0$.

**$B_{2,0}^0$:** The open subvariety of $B_2^0$ such that for all $i, j$, the line $m_{ij}$ has contact of order 2 with $Q$ at $u_i$ and its point of tangency with $C$ is not a node of $Q$.

**$B_{2,1}^0$:** The closed subvariety of $B_2^0$ parametrizing curves such that for some $i, j$, the line $m_{ij}$ has contact of order 4 with $Q$ at $u_i$.

**${B_{2,1}^0}''$:** The closed subvariety of $B_2^0$ such that $Q$ is the union of two lines and a nodal cubic.

**${B_{2,1}^0}'$:** The closed subvariety of $B_2^0 \setminus {B_{2,1}^0}''$ such that, for some $i, j$, the point of tangency of $m_{ij}$ with $C$ is a node of $Q$ (i.e., is on $l_1$ or $l_2$).

Note that for any stratum $B_{i,j}^0$, ${B_{i,j}^0}'$, ${B_{2,1}^0}''$ above, the codimension of $B_{i,j}^0$ (resp. ${B_{i,j}^0}'$, resp. ${B_{2,1}^0}''$) in $B_i$ is equal to $j$.

**Theorem 4.2.** *The Chow groups of the varieties $B_{i,j}^0$, ${B_{i,j}^0}'$, ${B_{2,1}^0}''$, $B_3^0$ and $B_5$ are trivial.*

*Proof:* The first two parts of this proof proceed along the same lines as the proof of Theorem 3.1. We will therefore omit the details that are similar to those in that proof.

In this preliminary part of the proof, we will let $P$ be any of the varieties $B_{1,j}^0$, ${B_{1,j}^0}'$. Let $S(P)$ be the quartic Del Pezzo surface (defined up to projective equivalence) such that for any $X$ with $m_X \in P$, the curve $X$ embeds in $S(P)$ (i.e., $S(P) = S(m_i)$ for all $X$ with moduli point in $P$). Fix a quadric $q_0$ of rank 3 and a projective embedding of $S(P)$ in $q_0$ such that $q_0$ corresponds to $u_i$ in $m_i$. By [3] page 365, all the quadrics of rank 4 parametrized by $l$ have the same vertex (which is not on $X$). By continuity, this vertex is contained in $Sing(q_0)$ and it is not on $S(P)$ because $X$ is the intersection of $S(P)$ with the quadrics parametrized by $l$. Fix a general point $p_0 \in Sing(q_0)$.

Since the embedding of $X$ in $\mathbb{P}^4$ is defined up to projective equivalence, we can suppose that $p_0$ is the common vertex of the quadrics parametrized by $l$. So, if we let $\mu : \mathbb{P}^4 \longrightarrow \mathbb{P}^3$ be the projection from $p_0$, for any $X$ such that $m_X \in P$, the curve $X$ is the complete intersection of $S(P)$ with $\mu^*q$ for some quadric $q$ in $\mathbb{P}^3$ such that $\mu^*q \neq q_0$. Let $U(P)$ be the open subvariety of $L := \frac{\mu^*|\mathcal{O}_{\mathbb{P}^3}(2)|}{\{q_0\}}$ which parametrizes pencils of quadrics $\langle \mu^*q, q_0\rangle$ such that $\mu^*q \cap S(P)$ is a smooth canonical curve with moduli point in $P$. Below we describe (for each $P$) a linear subspace $L(P)$ of $L$ whose intersection with $U(P)$ has codimension 1 complement in $L(P)$ and maps onto $P$ in a finite way.

**$B^0_{1,0}$:** The surface $S(B^0_{1,0})$ is described in Lemma 2.3,10. The space $L(B^0_{1,0})$ is equal to $L$ (The automorphism group of $S(B^0_{1,0})$ is one-dimensional, it acts on $\mathbb{P}^4$ and leaves $Sing(q_0)$ globally invariant. The stabilizer of $p_0$ is finite since $p_0$ is general.)

**$B^0_{1,1}$:** The surface $S(B^0_{1,1})$ is described in Lemma 2.3,12. It contains two lines which intersect at its double point. Choose a general point $p$ on one of these lines. Let $L(B^0_{1,1})$ be the linear subspace of $L$ parametrizing pencils of quadrics through $p$.

**${B^0_{1,1}}'$:** The surface $S({B^0_{1,1}}')$ is described in Lemma 2.3,11. The space $L({B^0_{1,1}}')$ is defined in a way analogous to the previous case.

**$B^0_{1,2}$:** The surface $S(B^0_{1,2})$ is described in Lemma 2.3,13. It contains only one line. We claim that this line intersects $X$ in two distinct points which are fixed points for the bielliptic involution of $X$: Let $X \longrightarrow E$ be a morphism of degree 2 onto an elliptic curve and $s_1, \dots, s_8$ its ramification points. For $i \neq j$, the pencil $l_{ij}$ of quadrics in $\mathbb{P}^4$ containing $X$ and $\langle s_i + s_j\rangle$ is not contained in $Q$: Otherwise, by [3] page 365, all the quadrics parametrized by $l_{ij}$ are singular at some point, say $x_0$. The base locus $S(l_{ij})$ of $l_{ij}$ is then a cone with vertex $x_0$ over a quartic elliptic curve in $\mathbb{P}^3$. The curve $X$ maps onto this curve by a morphism of degree 2. So, since $X$ has only one bielliptic structure, this curve is isomorphic to $E$. Therefore all the lines in $S(l_{ij})$ (in particular $\langle s_i + s_j\rangle$) contain $p_0$. So $s_i$ and $s_j$ project to the same point on $E$: contradiction.

Now, $h^0(2s_i + 2s_j) = 2$ implies that $S(l_{ij})$ contains less than 16 lines and therefore is not smooth by Lemma 2.1. Hence, by Lemma 2.3, the line $l_{ij}$ is bitangent to $Q$. Now suppose for a moment that $Q$ is the union of a line $L$ and a smooth plane quartic $R$. We claim that $l_{ij}$ is bitangent to $R$. Indeed, $l_{ij}$ is either bitangent to $R$ or contains a node of $Q$ which corresponds to a quadric $q$ of rank 3. If $l_{ij}$ contains a node of $Q$, the configuration of double points of $X^{s_i s_j}$ is as in Lemma 2.3 case 8 or 9 or 11 or 13. So the vanishing theta-null cut by the ruling of $q$ is $|s_i + s_j + p_1' + p_1''|$. The quadric $q$ is the pull-back by $\mu : \mathbb{P}^4 \longrightarrow \mathbb{P}^3$ of a quadric in $\mathbb{P}^3$ since it belongs to $l$. So the vanishing theta-null cut by its ruling is the pull-back of a $g^1_2$ on $E$. It follows that we have $s_i + s_j + p_1' + p_1'' = 2s_i + 2s_j$, i.e., the configuration of double points of $X^{s_i s_j}$ is 2.3 case 11 or 13 and $l_{ij}$ is bitangent to $R$.

Since $R$ has 28 bitangents and there are 28 divisors $s_i + s_j$, we see that *these are the only divisors of degree 2 on $X$ which are not pullbacks from $E$ and verify $h^0(2s_i + 2s_j) = 2$*. So, for $R$ smooth, the line in $S(B^0_{1,2})$ intersects $X$ in $s_i$ and $s_j$ for some $i \neq j$. For $R$ not necessarily smooth, $m_i$ is a limit of bitangents of smooth quartics and by continuity the line in $S(B^0_{1,2})$ intersects $X$ in two points which are fixed points for the bielliptic involution of $X$. These are distinct since such points cannot come together when $X$ varies in a family unless $X$ becomes singular.

Now choose two general points on this line. Let $L(B^0_{1,2})$ be the linear subspace of $L$ parametrizing pencils of quadrics through these points.

**${B^0_{1,2}}'$:** The surface $S({B^0_{1,2}}')$ is described in Lemma 2.3,15. It contains two lines which intersect at one of its double points. Choose a general point on each line. Let $L({B^0_{1,2}}')$ be the linear subspace of $L$ parametrizing pencils of quadrics through these points.

Now let $P$ denote, momentarily, one of the varieties $B^0_{2,i}$ or ${B^0_{2,1}}'$. Again, we have a quartic Del Pezzo surface $S(P)$ such that every curve with moduli point in $P$ embeds in $S(P)$ and $S(P)$ is projectively isomorphic to the base locus of one of the lines $m_{ij}$. The surfaces $S(B^0_{2,0})$, $S(B^0_{2,1})$, $S({B^0_{2,1}}')$ are described respectively in 2.3 cases 8, 12 and 15. For each $P$, the surface $S(P)$ contains (exactly) two lines $k_1$ and $k_2$ which intersect (each other and) any other line in $S(P)$ only at the double points of $S(P)$ (these lines are $\langle s+t\rangle$ and $\langle p_1'+p_1''\rangle$ with the notation of 2.3). When we let the quartic $R$ in the case of $B^0_{1,2}$ degenerate to the union $C\cup l_2$, some of the bitangents to $R$ degenerate to the lines $m_{ij}$ with $1\le i\le 3$. Therefore, by continuity, each of the lines $k_1$ and $k_2$ in $S(P)$ intersects $X$ in two distinct points which are fixed points of the bielliptic involution of $X$ corresponding to $l_1$. In particular, the tangent lines to $X$ at these four points all meet in one point which is $\mathrm{Sing}(q_5)$ since $q_5$ is on $l_1$ with the notation of 2.3. This point is neither on $\mathrm{Sing}(q_0)$ nor on $S(P)$. It is, however, on the singular locus of the quadric of rank 3 which is the intersection of $l_1$ and $l_2$. It is also in the intersection of the two planes which are tangent to $S(P)$ along $k_1$ and $k_2$ respectively. We therefore choose a general point $x_0$ on the intersection of these two planes (this intersection is a line since the two planes are contained in the hyperplane tangent at $k_1\cap k_2$ to a quadric containing $S(P)$ which is smooth at $k_1\cap k_2$). We let $L$ be the linear system parametrizing quadrics of rank 3 with singular locus $\langle p_0,x_0\rangle$. Let $U(P)$ be the open subset of $L$ parametrizing quadrics such that their intersection with $S(P)$ is a smooth curve with moduli point in $P$.

$B^0_{2,0}$: The open subset $U(B^0_{2,0})$ maps onto $B^0_{2,0}$ in a finite way (the stabilizer of $x_0$ in the automorphism group of $S(B^0_{2,0})$ (which has dimension 1) is finite).

$B^0_{2,1}$, ${B^0_{2,1}}'$: Fix a general point $p$ on $k_1$. Let $L(B^0_{2,1})$ (resp. $L({B^0_{2,1}}')$) be the sublinear system of $L$ parametrizing quadrics through $p$. Then $U(B^0_{2,1})\cap L$ (resp. $U({B^0_{2,1}}')\cap L$) maps onto $B^0_{2,1}$ (resp. ${B^0_{2,1}}'$) in a finite way.

We prove the triviality of the Chow rings of ${B^0_{2,1}}''$, $B^0_3$ and $B_5$ by a different method. By [3] pages 316-317, 321, 324, 362, the (ramified) double cover $\widetilde{Q}\longrightarrow Q$, where $\widetilde{Q}$ parametrizes the rulings of the quadrics parametrized by $Q$, is ramified at all the nodes of $Q$ and the branches of $\widetilde{Q}$ at each node are not exchanged by the covering involution; furthermore, the Prym variety of this double cover is the jacobian of $X$ and the data of the double cover is equivalent to the data of $X$. Since all the components of $Q$ are rational for $m_X\in {B^0_{2,1}}''\cup B^0_3\cup B_5$, the double cover $\widetilde{Q}\longrightarrow Q$ is determined by $Q$ and so the data of $X$ is equivalent to the data of the nodal plane quintic $Q$. It follows that

1. ${B^0_{2,1}}''$ is isomorphic to the moduli space of nodal plane quintics which are unions of two lines and a nodal irreducible cubic. The Chow ring of this space is trivial: Fix two lines $l_1$ and $l_2$, a general point $r_i$ on $l_i$ and a general point $x$ in the plane. Then the space of quintics which are the union of $l_1$, $l_2$ and an irreducible cubic with a node at $x$ and passing through $r_1,r_2$ maps in a finite way onto ${B^0_{2,1}}''$.
2. $B^0_3$ is isomorphic to the moduli space of nodal plane quintics which are unions of three lines and an irreducible conic. Fix three general lines $l_1,l_2,l_3$ and two general points $x_1$ and $x_2$ on $l_1$ and $l_2$ respectively. The space of quintics with only nodes which are the union of $l_1,l_2,l_3$ and an irreducible conic through $x_1$ and $x_2$ maps in a finite way onto $B^0_3$.
3. $B_5$ is isomorphic to the moduli space of nodal plane quintics which are unions of five lines. By duality, this is isomorphic to the moduli space of 5 distinct points in $\mathbb{P}^2$ of which no three are collinear. Therefore the space of five *ordered* distinct points in the plane of which no three are collinear maps in a finite way onto $B_5$. By [6] page 114, the former is isomorphic to a smooth anticanonical Del Pezzo surface of degree 5 minus all of its lines: this has trivial Chow ring. Q.E.D.

Instead of computing the classes of the closures of the above subvarieties of $B$, we deduce from the above theorem that $A_*(B)$ is generated by pullbacks of cycles from the moduli space $\mathcal{Q}$ of plane quintics which contain a line and have only nodes as singularities.

Let $\tilde{\mathcal{Q}} \subset (\mathbb{P}^2)^* \times \mathcal{Q}$ be the space parametrizing pairs $(l, Q)$ such that the line $l$ is contained in the plane quintic $Q$. The morphism $\tilde{\mathcal{Q}} \longrightarrow \overline{\mathcal{M}}_3$ sending $(l, Q)$ to the plane quartic curve $Q \setminus l$ allows us to identify $\tilde{\mathcal{Q}}$ with an open subset of the projectivization of the Hodge bundle over $\overline{\mathcal{M}}_3$. The image $\widetilde{\mathcal{M}}_3$ of $\tilde{\mathcal{Q}} \longrightarrow \overline{\mathcal{M}}_3$ is the subvariety of $\overline{\mathcal{M}}_3$ parametrizing stable curves $C$ which *embed* into $\mathbb{P}^2 = |\omega_C|^*$ as plane quartics, its fiber at a given plane quartic $C$ is the set of lines in $|\omega_C|^*$ which are transverse to the image of $C$, i.e., the complement in $|\omega_C|$ of the full dual variety of the image of $C$.

So, if we denote by $\xi$ the first Chern class of the tautological line bundle on $\tilde{\mathcal{Q}}$ (as an open subset of a projective bundle on $\widetilde{\mathcal{M}}_3$), the Chow ring (well-defined because $\tilde{\mathcal{Q}}$ is an open subset of a projective bundle over $\widetilde{\mathcal{M}}_3$) of $\tilde{\mathcal{Q}}$ is generated by the Chow ring of $\widetilde{\mathcal{M}}_3$ and $\xi$. Now, by the above, the class of the complement of $\tilde{\mathcal{Q}}$ in the projective Hodge bundle is the sum of $12\xi$ and the pull-back of a class on $\widetilde{\mathcal{M}}_3$. Hence (since our Chow rings are with rational coefficients), the Chow ring of $\tilde{\mathcal{Q}}$ is isomorphic to $A_*(\widetilde{\mathcal{M}}_3)$. The Chow ring of $\overline{\mathcal{M}}_3$ is computed in [9], it is:

$$\mathbb{Q}[\lambda, \delta_0, \delta_1, \kappa_2]/I,$$

where we also denote by $\lambda$ and $\kappa_2$ Mumford's tautological classes for $\overline{\mathcal{M}}_3$, $\delta_0$ is the class of the boundary component $\Delta_0$ of $\overline{\mathcal{M}}_3$ whose general elements are irreducible curves with one node, $\delta_1$ is the class of the boundary component $\Delta_1$ of $\overline{\mathcal{M}}_3$ whose general elements are unions of smooth curves of genus 2 and smooth elliptic curves meeting at 1 point, and $I$ is an ideal. This result has been proved in characteristic zero. However, the proofs of the triviality of the Chow rings of the strata for $\overline{\mathcal{M}}_3$ are valid in characteristic $\neq 2$ and 3 and the relations (and computations of classes of subvarieties as combinations of tautological classes) proved in [9] remain valid in positive characteristic by continuity. Since the elements of $\Delta_1$ cannot be embedded as canonical plane quartics, we deduce that the restriction of $\delta_1$ to $\widetilde{\mathcal{M}}_3$ is 0. By [13] section 6, the Q-class (see [17] section 3) of the hyperelliptic locus in $\overline{\mathcal{M}}_3$ is equal to $9\lambda - \delta_0 - 3\delta_1$, therefore, the (usual) class of the hyperelliptic locus in $\widetilde{\mathcal{M}}_3$ is $2(9\lambda - \delta_0)$. Since hyperelliptic curves cannot be embedded as canonical plane quartics either, it follows that $9\lambda - \delta_0$ is zero in $\widetilde{\mathcal{M}}_3$. Next we deduce from the formulas in [9] pages 343, 366 and 368 (valid by continuity in positive characteristic) that the restriction of $\kappa_2$ to $\widetilde{\mathcal{M}}_3$ is a multiple of $\delta_0^2$. So $A_*(\widetilde{\mathcal{M}}_3)$, and hence $A_*(\tilde{\mathcal{Q}})$, is generated by $\delta_0$ ($\delta_0$ is nonzero because it is independent of $\lambda$ and $\delta_1$ in $\overline{\mathcal{M}}_3$ and the general elements of $\Delta_0$ *can* be embedded as plane quartics; however, it follows from relation 1) on page 389 of [9] that $\delta_0^3 = 0$ in $A_*(\widetilde{\mathcal{M}}_3)$).

Since the morphism $\tilde{\mathcal{Q}} \longrightarrow \mathcal{Q}$ is finite (and an isomorphism on an open set which meets the inverse image of $\Delta_0$ in $\tilde{\mathcal{Q}}$), we obtain that $A_*(\mathcal{Q})$ is also generated by $\delta_0$ (and $\delta_0 \neq 0$).

Let $B'$ be the locus of bielliptic curves of genus 5 with at least 5 vanishing theta-nulls. By the above and Corollary 3.3, the classes $[B]_V$, $[B']_V$ and $[B']_V^2$ are the only possibly nonzero classes in the Chow ring of $V = U \cup B$. We have

**Proposition 4.3.** *The class of $B'$ in $V$ is a combination of tautological classes.*

*Proof:* Let $X \longrightarrow E$ and $s_1, \ldots, s_8$ be as in the proof of Theorem 4.1 and suppose that $X$ has only one bielliptic structure. Suppose that the quartic $R$ in $Q$ has a node $q_0$ and $Q$ is general for this property. Then there are 6 lines $n_1, ..., n_6$ through the node of $R$ which are elsewhere tangent to $R$. As in the proof of Theorem 4.2, for the cases of $B^0_{1,2}$ and of the strata in $B^0_2$, for each $i \in \{1, \ldots, 6\}$, there are two lines $\langle s_{k(i)} + s_{h(i)} \rangle$ and $\langle s_{m(i)} + s_{n(i)} \rangle$ which are contained in the base locus $S(n_i)$ of $n_i$ so that $|s_{k(i)} + s_{h(i)} + s_{m(i)} + s_{n(i)}|$ is the vanishing theta-null on $X$ cut by the ruling of $q_0$. Now construct the subvariety $\mathcal{P}'''$ of $\mathcal{C}^7$ which is the closure of the subvariety parametrizing 4-tuples $(s_1, s_2, s_3, s_4)$ of distinct points such that $h^0(2s_1 + 2s_i) \geq 2$ for $i = 2, 3, 4$, $h^0(4s_1 + 2s_2) \geq 3$ and $h^0(s_1 + s_2 + s_3 + s_4) \geq 2$. Then, as in the proof of Theorem 4.1, the pushforward of $\mathcal{P}'''$ to $V$ is a combination of tautological classes. The rest of the proof is analogous to the proof of Theorem 4.1. Q.E.D.

## 5. The Chow groups of $T$ and $W$

The class of $\overline{T}$ in $\mathcal{M}_5$ was computed in [13] (Theorem 3 page 53, the proof is valid in characteristic $\neq 2$ since the computation is as in the proof of Theorem 3.2 above: the variety $\overline{T}$ is the image in $W := T \cup U \cup B$ of the locus in $\mathcal{C}^2_W$ defined by the condition $h^0(2s + t) \geq 2$) and is a positive multiple of $\lambda$. Below we write a stratification for $T$ and show that the classes in $W$ of the closures of the strata are combinations of tautological classes. Recall (see Section 1 above) that $\overline{T} \cap B = \emptyset$ and that a non-hyperelliptic trigonal curve has a unique $g^1_3$.

By, e.g. [20] page 175, every smooth canonical trigonal curve embeds in a smooth rational normal scroll. In $\mathbb{P}^4$ there is only one such scroll $D$ which has degree 3 and is the projection of the Veronese surface in $\mathbb{P}^5$ from a point on it. The point from which one projects blows up to a smooth rational curve $E$ of self-intersection $-1$ on $D$. The scroll $D$ is isomorphic to the projective bundle associated to the vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ on $\mathbb{P}^1$ and $E$ is the only section of $D$ over $\mathbb{P}^1$ with self-intersection $-1$. The curve $E$ and the fibers of $D$ over $\mathbb{P}^1$ are lines in $\mathbb{P}^4$ and $D$ is embedded in $\mathbb{P}^4$ by the line bundle $\mathcal{O}_D(E + 2F)$ where $F$ is a fiber. Every smooth trigonal canonical curve $X$ of genus 5 is the residual intersection of $D$ with a cubic hypersurface containing some fiber of $D$ over $\mathbb{P}^1$. So $X$ is a member of the linear system $|\mathcal{O}_D(3(E + 2F) - F)| = |\mathcal{O}_D(3E + 5F)|$. The automorphism group of $D$ has dimension 6 and acts on this linear system. The fibers of $D$ cut on $X$ the divisors of its $g^1_3$ and $|K_X| = |\mathcal{O}_X(1)| = |(E + 2F).X|$. It follows that $|E.X| = |K_X - 2g^1_3|$ and, in particular, $|E.X|$ has degree 2. Choose three general fibers $F_0, F_1$ and $F_2$ with general points $p_i$ on $F_i$ and let $e_i$ be the point of intersection of $F_i$ with $E$. We define the following strata in $T$:

- $T_0$: The image in $T$ of the locally closed subset of $|\mathcal{O}_D(3E + 5F)|$ parametrizing smooth curves whose $g^1_3$ has no double ramification points, no ramification points on $E$ and has ramification points at $p_0$, $p_1$ and $p_2$.
- $T_1$: The image in $T$ of the locally closed subset of $|\mathcal{O}_D(3E + 5F)|$ parametrizing smooth curves whose $g^1_3$ has ramifications point at $e_0$, $p_1$ and $p_2$, has no other ramification point on $E$ and no double ramification points.
- $T'_1$: The image in $T$ of the locally closed subset of $|\mathcal{O}_D(3E + 5F)|$ parametrizing smooth curves whose $g^1_3$ has no ramification points on $E$, has exactly one double ramification point which is $p_0$ and has (simple) ramification points at $p_1$ and $p_2$.
- $T_2$: The image in $T$ of the locally closed subset of $|\mathcal{O}_D(3E + 5F)|$ parametrizing smooth curves whose $g^1_3$ has no double ramification points and has ramification points at $e_0$, $e_1$ and $p_2$.
- $T'_2$: The image in $T$ of the locally closed subset of $|\mathcal{O}_D(3E + 5F)|$ parametrizing smooth curves whose $g^1_3$ has exactly one double ramification point which is $p_2$, has ramification points at $e_0$ and $p_1$ and no other ramification point on $E$.
- $T''_2$: The image in $T$ of the locally closed subset of $|\mathcal{O}_D(3E + 5F)|$ parametrizing smooth curves whose $g^1_3$ has exactly one double ramification point which is at $e_0$, has no other ramification point on $E$ and has ramification points at $p_1$ and $p_2$.
- $T'''_2$: The image in $T$ of the locally closed subset of $|\mathcal{O}_D(3E + 5F)|$ parametrizing smooth curves whose $g^1_3$ has no ramification points on $E$, has a (simple or double) ramification point at $p_0$ and double ramification points at $p_1$ and $p_2$.

$T_3$: The image in $T$ of the locally closed subset of $|\mathcal{O}_D(3E+5F)|$ parametrizing smooth curves whose $g^1_3$ has a double ramification point at $p_2$ and ramification points at $e_0$ and $e_1$.

$T'_3$: The image in $T$ of the locally closed subset of $|\mathcal{O}_D(3E+5F)|$ parametrizing smooth curves whose $g^1_3$ has exactly one double ramification point which is at $e_0$ and has (simple) ramification points at $e_1$ and $p_2$.

$T''_3$: The image in $T$ of the locally closed subset of $|\mathcal{O}_D(3E+5F)|$ parametrizing smooth curves whose $g^1_3$ has a ramification point at $e_0$, no other ramification point on $E$ and has double ramification points at $p_1$ and $p_2$.

$T'''_3$: The image in $T$ of the locally closed subset of $|\mathcal{O}_D(3E+5F)|$ parametrizing smooth curves whose $g^1_3$ has double ramification points at $e_0$ and $p_2$, a ramification point at $p_1$ and no other ramification point on $E$.

$T_4$: The image in $T$ of the locally closed subset of $|\mathcal{O}_D(3E+5F)|$ parametrizing smooth curves whose $g^1_3$ has double ramification points at $e_0$ and $e_1$ and a ramification point at $p_2$.

We have

**Theorem 5.1.** *The Chow groups of $T_i, T'_i, T''_i, T'''_i$ are trivial. The classes of the closures $\overline{T}_i, \overline{T}'_i, \overline{T}''_i, \overline{T}'''_i$ of the above strata in $W$ are combinations of tautological classes.*

*Proof:* Every locally closed subvariety above which maps onto one of our strata is an open dense subset with codimension 1 complement of a linear subsystem of $|\mathcal{O}_D(3E+5F)|$. Therefore it has trivial Chow ring. The proof of the first part is now analogous to the proof of Theorem 3.1. For the second part we proceed as in the proof of Theorem 3.2 (as in that proof, to apply determinantal formulas, we implicitly replace $W$ by a smooth finite cover and we replace all the subvarieties of $W$ by their inverse images).

For $X$ trigonal, put $|K_X - 2g^1_3| = \{p+q\}$.

$\overline{T}_1$: The moduli point $m_X$ is in $\overline{T}_1$ if and only if one of the points $p$ or $q$, say $p$, is a ramification point of $g^1_3$. Consider the subvariety of $\mathcal{C}^4_W$ defined by the conditions

$$h^0(2t+s) \geq 2 \text{ and } h^0(2t+s+p+q) \geq 3$$

This subvariety can be expressed, as in the proof of Theorem 3.2 as the degeneracy locus of a map of vector bundles with respect to some flag and it has the right codimension so we can compute its class using determinantal formulas. Since $T_1$ is the pushforward to $W$ of the intersection of this subvariety with the diagonal $\Delta_{1,3}$, we deduce, as in Theorem 3.2, that $[\overline{T}_1]_W$ is a combination of tautological classes.

$\overline{T}'_1$: This is the pushforward to $W$ of the subvariety of $\mathcal{C}_W$ defined by the condition $h^0(3t) \geq 2$ which is a degeneracy locus of a map of vector bundles and has the right codimension. So the class of $\overline{T}'_1$ is a combination of tautological classes.

$\overline{T}_2$: The moduli point $m_X$ is in $\overline{T}_2$ if and only if $p$ and $q$ are both ramification points of $g^1_3$. Consider the subvariety $P$ of $\mathcal{C}^6_W$ defined by the conditions

$$h^0(2t+s) \geq 2, \ h^0(2t+s+p+q) \geq 3$$

$$h^0(2t+s+p+q+2r) \geq 4, \ h^0(2t+s+p+q+2r+u) \geq 5$$

As before $P$ is a degeneracy locus of the right dimension. Its intersection with $\Delta_{1,3} \cap \Delta_{4,5}$ pushes forward to $\overline{T}_2$ (we cannot have $p=q$ and $p$ is a ramification point of the $g^1_3$, since then $X$ would be singular (tangent to $E$ and a fiber at the same point)) so, as before, the class of $\overline{T}_2$ is a combination of tautological classes.

$\overline{T}'_2$: The intersection of the variety $P$ above with $\Delta_{1,2} \cap \Delta_{4,5}$ pushes forward to $\overline{T}'_2$ and has the correct codimension. We conclude as before.

$\overline{T}''_2$: The intersection of $P$ with $\Delta_{1,2,3}$ pushes forward to $\overline{T}''_2$ and has the correct dimension.

$\overline{T}'''_2$: Consider the subvariety of $\mathcal{C}^6_W$ defined by the conditions

$$h^0(2t+s) \geq 2,\ h^0(2t+s+p+q) \geq 3$$

The intersection $R$ of this subvariety with $\Delta_{1,5} \cap \Delta_{2,6}$ is contained in $P$ and its class is a combination of the diagonals and the $K_i$'s (see the proof of Theorem 3.2). Therefore the class of the closure $P'$ of $P \setminus R$ is also a combination of diagonals and $K_i$'s and its pushforward to $W$ is a combination of tautological classes. It is immediately seen that $\overline{T}'''_2$ is the pushforward of $P' \cap \Delta_{1,2} \cap \Delta_{5,6}$ to $W$.

$\overline{T}_3$: We remark that $\overline{T}_3 \cup \overline{T}'_3 = \overline{T}_2 \cap \overline{T}'_1$. Therefore a linear combination with positive coefficients of the classes of $\overline{T}_3$ and $\overline{T}'_3$ is a combination of tautological classes. So the fact that the class of $\overline{T}_3$ is a combination of tautological classes follows from the same fact for $\overline{T}'_3$ which we prove below.

$\overline{T}'_3$: The variety $\overline{T}'_3$ is the pushforward to $W$ of the intersection $P \cap \Delta_{1,2,3} \cap \Delta_{4,5}$.

$\overline{T}''_3$: As in the case of $\overline{T}_3$, we remark that $\overline{T}''_3 \cup \overline{T}'''_3 = \overline{T}'_2 \cap \overline{T}_1$.

$\overline{T}'''_3$: The variety $\overline{T}'''_3$ is the pushforward to $W$ of the intersection $P' \cap \Delta_{1,2,3} \cap \Delta_{5,6}$.

$\overline{T}_4$: The variety $\overline{T}_4$ is the pushforward to $W$ of the intersection $P \cap \Delta_{1,2,3} \cap \Delta_{4,5,6}$.

Q.E.D.

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