A PRYM CONSTRUCTION FOR THE COHOMOLOGY OF A CUBIC HYPERSURFACE

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Introduction

Fano studied the variety of lines on a cubic hypersurface with a finite number of singular points. The variety parametrizing linear spaces of given dimension in a projective variety X is now called a Fano variety. Subvarieties of a Fano variety can be defined using various incidence relations. Such varieties are studied to help understand the geometric properties of X and for their own sake. For instance, the proofs of the irrationality of a smooth cubic threefold X and of the Torelli theorem for X by Clemens and Griffiths use varieties of lines in the cubic.

Suppose that X is a smooth cubic hypersurface in $\mathbb{P}^4$ and let F be the Fano variety of lines in X. By [6] (Lemma 7.7 page 312) the variety F is a smooth surface. Let us fix a general line l in X, corresponding to a general element of F, and let D_l be the variety of lines in X incident to l. The blow up X_l of X along l has the structure of a conic-bundle over $\mathbb{P}^2$ and its discriminant curve is a smooth plane quintic Q_l:

$$\begin{tikzcd} & X_l \arrow[d] \\ Q_l \arrow[r, phantom, "\subset"] & \mathbb{P}^2 . \end{tikzcd}$$

The curve D_l is an étale double cover of Q_l.

In a first proof of the irrationality of X, Clemens and Griffiths use the canonical isomorphism between the Albanese variety of F and the intermediate jacobian of X (see [6], Theorem 11.19 page 334). In a second proof they use the canonical isomorphism (due to Mumford, see [6], Appendix C) between the intermediate jacobian of X and the Prym variety of the (étale) double cover $D_l \to Q_l$. More generally, Mumford's result says that this isomorphism holds for a conic bundle X over $\mathbb{P}^2$ with discriminant curve Q_l and double cover D_l parametrizing the components of the singular conics parametrized by Q_l. Beauville generalized this isomorphism to the case where X is an odd-dimensional quadric bundle over $\mathbb{P}^2$ with discriminant curve Q_l and double cover D_l parametrizing the rulings of the quadrics parametrized by Q_l (see [1]).

In this paper we "generalize" the isomorphism between the intermediate jacobian of X and the Prym variety of $D_l \to Q_l$ to the cohomology of higher dimensional cubic hypersurfaces. On the way we also obtain some results about the Fano variety $\mathcal{P}$ of planes in X.

A principally polarized abelian variety A is the Prym variety of a double cover of curves $\pi : \widetilde{C} \to C$ if there is an exact sequence

$$0 \longrightarrow \pi^* JC \longrightarrow J\widetilde{C} \longrightarrow A \longrightarrow 0$$

and, under the transpose of $J\widetilde{C} \twoheadrightarrow A$, the principal polarization of $J\widetilde{C}$ pulls back to twice the principal polarization of A. The generalization that we have in mind would say that a polarized Hodge structure H is the Prym Hodge structure of two polarized Hodge structures $H_1 \subset H_2$ if there is an involution $i : H_2 \to H_2$ and a surjective morphism of Hodge structures $\psi : H_2 \twoheadrightarrow H$ such that i is a morphism of Hodge structures of type $(0,0)$, the kernel of ψ is the i-invariant part of H_2 which is equal to H_1 and such that for any two i-anti-invariant elements a, b of H_2 we have $\psi(a).\psi(b) = -2a.b$ where "." denotes the polarizations (see [1] page 334). In our case H will be the primitive cohomology of a cubic hypersurface and H_1 and H_2 will be the "primitive" cohomologies of (partial) desingularizations of Q_l and D_l.

From now on let X be a smooth cubic hypersurface in $\mathbb{P}^n$. For a general line $l \subset X$, we define X_l to be the blow up of X along l. Then X_l is a conic bundle over $\mathbb{P}^{n-2}$ and we define Q_l to be its discriminant variety:

$$\begin{tikzcd} & X_l \arrow[d] \\ Q_l \arrow[r, phantom, "\subset"] & \mathbb{P}^{n-2}. \end{tikzcd}$$

For $n \geq 5$ the variety Q_l is singular. It parametrizes the singular or higher-dimensional fibers of $X_l \to \mathbb{P}^{n-2}$ and it can be thought of as the variety parametrizing planes in $\mathbb{P}^n$ which contain l and, either are contained in X or, whose intersection with X is a union of three (possibly equal) lines. We define D_l to be the variety of lines in X incident to l. Then D_l admits a rational map of degree 2 to Q_l and the varieties D_l and Q_l have dimension $n-3$. It is proved in [15], page 590, that D_l is smooth and its map to Q_l is a morphism for $n = 5$ and l general. We show that, for $n \geq 6$, the variety D_l is always singular and the rational map $D_l \to Q_l$ is never a morphism. We define a natural desingularization S_l of D_l such that the rational map $D_l \to Q_l$ lifts to a morphism $S_l \to Q_l$. However, for $n \geq 8$, the morphism is not finite. So we define natural blow-ups S'_l and Q'_l of S_l and Q_l such that the morphism $S_l \to Q_l$ lifts to a double cover $S'_l \to Q'_l$. The varieties S_l and S'_l naturally parametrize lines in blow-ups of X_l so that we have Abel-Jacobi maps $\psi : H^{n-3}(S_l, \mathbb{Z}) \to H^{n-1}(X, \mathbb{Z})$ and $\psi' : H^{n-3}(S'_l, \mathbb{Z}) \to H^{n-1}(X, \mathbb{Z})$. Our main results are as follows

Lemma 1. *The Abel-Jacobi maps*

$$\psi : H^{n-3}(S_l, \mathbb{Z}) \longrightarrow H^{n-1}(X, \mathbb{Z})$$

and

$$\psi' : H^{n-3}(S'_l, \mathbb{Z}) \longrightarrow H^{n-1}(X, \mathbb{Z})$$

are onto.

The involution $i_l : S'_l \to S'_l$ of the double cover $S'_l \to Q'_l$ induces an involution $i : H^{n-3}(S'_l, \mathbb{Z}) \to H^{n-3}(S'_l, \mathbb{Z})$ whose invariant subgroup is $H^{n-3}(Q'_l, \mathbb{Z})$. However, the Prym construction only works for "primitive" cohomologies (see Definition 5.7 below). Denote the primitive part of each cohomology group H by H^0. We need to show that for any two i-anti-invariant elements a, b of $H^{n-3}(S'_l, \mathbb{Z})^0$, we have $\psi'(a).\psi'(b) = -2a.b$. This follows from (see 5.9)

Theorem 2. *Let a and b be two elements of $H^{n-3}(S'_l, \mathbb{Z})^0$. Then*

$$\psi'(a).\psi'(b) = a.i_l^* b - a.b\,.$$

We use this theorem to prove

Theorem 3. *The Abel-Jacobi map*

$$\psi'^0 : H^{n-3}(S'_l, \mathbb{Z})^0 \longrightarrow H^{n-1}(X, \mathbb{Z})^0$$

is onto with kernel equal to the image of $H^{n-3}(Q'_l, \mathbb{Z})^0$ in $H^{n-3}(S'_l, \mathbb{Z})^0$.

This finishes the Prym construction.

We now discuss two applications of the above Prym construction. The first has to do with the Hodge conjectures. The general Hodge conjecture $GHC(X, m, p)$ as stated in [14] page 166 is the following

> $GHC(X, m, p)$: For every $\mathbb{Q}$-Hodge substructure V of $H^m(X, \mathbb{Q})$ with level $\leq m - 2p$, there exists a subvariety Z of X of codimension p such that V is contained in the image of the Gysin map $H^{m-2p}(\widetilde{Z}, \mathbb{Q}) \to H^m(X, \mathbb{Q})$ where $\widetilde{Z}$ is a desingularization of Z.

It is proved in [14], Proposition 2.6, that $GHC(Y, m, 1)$ holds for all uniruled smooth varieties Y of dimension m. Our Lemma 1 gives a geometric proof of GHC(X,n-1,1) for a smooth cubic hypersurface X in $\mathbb{P}^n$: the full cohomology $H^{n-1}(X, \mathbb{Z})$ is supported on the subvariety Z which is the union of all the lines in X incident to l.

The second application is (see Section 6)

Theorem 4. *The Abel-Jacobi map* $H^{n-1}(X,\mathbb{Z})^0 \xrightarrow{\phi} H^{n-3}(F,\mathbb{Z})^0$ *is an isomorphism of Hodge structures.*

This was proved for cubic threefolds by Clemens and Griffiths ([6] Theorem 11.19 page 334), for cubic fourfolds by Beauville and Donagi ([4]), for higher-dimensional cubic hypersurfaces by Shimada ([13], Theorem page 703 and Proposition 4 page 716).

An immediate consequence of Theorem 4 and Lemma 1 is

Corollary 5. *The push-forward* $H_{n-3}(S_l,\mathbb{Z}) \to H_{n-3}(F,\mathbb{Z})$ *is surjective.*

This fact was not known for $n \geq 5$.

We now describe our results in slightly greater detail. In Section 1 we prove that, for $n \geq 6$ and l general, the singular locus of D_l is $\{l\} \subset D_l$. Also, for $n \geq 6$, the natural map $D_l \to Q_l$ sending a line l' to the plane spanned by l and l' is only a rational map. In Section 2, we prove that the variety S_l parametrizing lines in the fibers of the conic bundle $X_l \to \mathbb{P}^{n-2}$ is a small desingularization of D_l which admits a *morphism* of generic degree 2 to Q_l. We show that S_l can also be defined as a subvariety of the product of Grassmannians of lines and planes in $\mathbb{P}^n$. For the generalized Prym construction we need a finite morphism of degree 2 to Q_l and the morphism $S_l \to Q_l$ is not finite for $n \geq 8$. It fails to be finite at the points of Q_l parametrizing planes contained in X (and containing l). Let T_l denote the variety parametrizing planes in X which contain l. Since $\mathbb{P}^{n-2}$ parametrizes the planes in $\mathbb{P}^n$ which contain l, the variety T_l is naturally a subvariety of $\mathbb{P}^{n-2}$ and in fact is contained in Q_l:

$$\begin{tikzcd}
 & & X_l \arrow[d] \\
T_l \subset Q_l & \subset & \mathbb{P}^{n-2}.
\end{tikzcd}$$

In Section 3 we prove that for l general T_l is a smooth complete intersection of the expected dimension $n-8$ in $\mathbb{P}^{n-2}$. For this we analyse the structure of the Fano variety $\mathcal{P}$ of planes in X. We prove that $\mathcal{P}$ is always of the expected dimension $3n-16$ and determine its singular locus. It is proved in [5] Theorem 4.1 page 33 or [7] Théorème 2.1 that $\mathcal{P}$ is connected for $n \geq 6$. We prove that $\mathcal{P}$ is irreducible for $n \geq 8$. In Section 4 we blow up $X_l \to \mathbb{P}^{n-2}$ along T_l and its inverse image in X_l to obtain $X_l' \to \mathbb{P}^{n-2'}$. The discriminant hypersurface of this conic-bundle is the blow-up Q_l' of Q_l along

T_l:

$$\begin{tikzcd} & X'_l \arrow[d] \\ Q'_l \subset & \mathbb{P}^{n-2'} . \end{tikzcd}$$

The variety S'_l is then defined as the variety of lines in the fibers of the conic bundle $X'_l \rightarrow \mathbb{P}^{n-2'}$. We prove that the rational involution acting in the fibers of $S_l \rightarrow Q_l$ lifts to a regular involution $i_l : S'_l \rightarrow S'_l$ and the quotient of S'_l by i_l is Q'_l. We also prove that S'_l is the blow up of S_l along the inverse image of T_l and the ramification locus R'_l of $S'_l \rightarrow Q'_l$ is smooth of codimension 2 and is an ordinary double locus for Q'_l. In Section 5 we prove Lemma 1, Theorem 2 and Theorem 3. We also prove some results about the rational cohomology ring of S_l: we prove that, except in the middle degree, this rational cohomology ring is generated by $H^2(S_l, \mathbb{Q})$ which, for $n \geq 6$, is generated by the inverse images h and σ_1 of the hyperplane classes of Q_l and D_l (the hyperplane class of D_l is the restriction of the hyperplane class of the Grassmannian of lines in $\mathbb{P}^n$). For $n = 5$, the space $H^2(S_l, \mathbb{Q})$ is the direct sum of its primitive part and $\mathbb{Q}h \oplus \mathbb{Q}\sigma_1$. In Section 6 we prove Theorem 4.

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NOTATION AND CONVENTIONS

The symbol n will always denote an integer ≥ 5.

For all positive integers k and l, we denote by $G(k, l)$ the Grassmannian of k-dimensional vector spaces in $\mathbb{C}^l$. For any vector space or vector bundle W, we denote by $\mathbb{P}(W)$ the projective space of lines in (the fibers of) W with its usual scheme structure.

For all cohomology vector spaces $H^i(Y, .)$ of a variety Y, we will denote by $h^i(Y, .)$ the dimension of $H^i(Y, .)$. For a point $y \in Y$, we denote by T_yY the Zariski tangent space to Y at y. If we are given an embedding $Y \subset \mathbb{P}^m = (\mathbb{C}^{m+1} \setminus \{0\})/\mathbb{C}^*$, we denote by T'_yY the inverse image of T_yY by the map $\mathbb{C}^{m+1} \twoheadrightarrow \mathbb{C}^{m+1}/\mathbb{C}v = T_y\mathbb{P}^m$ where v is a nonzero vector in $\mathbb{C}^{m+1}$ mapping to y. We call $\mathbb{P}(T'_yY)$ the projective tangent space to Y at y.

For any subsets or subschemes $Y_1, ..., Y_m$ of a projective space $\mathbb{P}^d$ (resp. affine space $\mathbb{C}^d$), we denote by $\langle Y_1, ..., Y_m \rangle$ the smallest linear subspace of $\mathbb{P}^d$ (resp. $\mathbb{C}^d$) containing $Y_1, ..., Y_m$.

For a subscheme Y_1 of a scheme Y_2, we denote by N_{Y_1/Y_2} the normal sheaf to Y_1 in Y_2.

For a global section s of a sheaf $\mathcal{F}$ on a scheme Y, we denote by $Z(s)$ the scheme of zeros of s in Y.

1. THE VARIETY D_l OF LINES INCIDENT TO l

For a smooth cubic hypersurface $X \subset \mathbb{P}^n$ of equation G, we let $\delta : \mathbb{P}^n \to (\mathbb{P}^n)^*$ be the dual morphism of X. In terms of a system of projective coordinates $\{x_0, ..., x_n\}$ on $\mathbb{P}^n$, the morphism δ is given by

$$\delta(x_0, ..., x_n) = (\partial_0 G(x_0, ..., x_n), ..., \partial_n G(x_0, ..., x_n))$$

where $\partial_i = \frac{\partial}{\partial x_i}$.

Let $l \subset X$ be a line. Following [6] (page 307 Definition 6.6, Lemma 6.7 and page 310 Proposition 6.19), we make the definition:

Definition 1.1. 1. *The line l is of first type if the normal bundle to l in X is isomorphic to $\mathcal{O}_l^{\oplus 2} \oplus \mathcal{O}_l(1)^{\oplus(n-4)}$. Equivalently, the intersection $\mathbb{T}_l$ of the projective tangent spaces to X along l is a linear subspace of $\mathbb{P}^n$ of dimension $n-3$. Equivalently, the dual morphism δ maps l isomorphically onto a conic in $(\mathbb{P}^n)^*$, i.e., the restriction map $\langle \partial_0 G, ..., \partial_n G\rangle \to H^0(l, \mathcal{O}_l(2))$ is onto where $\langle \partial_0 G, ..., \partial_n G\rangle$ is the span of $\partial_0 G, ..., \partial_n G$ in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$.*

2. *The line l is of second type if the normal bundle to l in X is isomorphic to $\mathcal{O}_l(-1) \oplus \mathcal{O}_l(1)^{\oplus(n-3)}$. Equivalently, the space $\mathbb{T}_l$ is a linear subspace of $\mathbb{P}^n$ of dimension $n-2$. Equivalently, the dual morphism δ has degree 2 on l and maps l onto a line in $(\mathbb{P}^n)^*$, i.e., the restriction map $\langle \partial_0 G, ..., \partial_n G\rangle \to H^0(l, \mathcal{O}_l(2))$ has rank 2.*

By [6] (Lemma 7.7 page 312), the variety F of lines in X is smooth of dimension $2(n-3)$. An easy dimension count shows that the dimension of D_l is at least $n-3$ for any $l \in F$. Suppose that l is of first type. We have

Lemma 1.2. *Let $l' \in D_l$ be distinct from l. If l' is of first type or if l' is of second type and l is* not *contained in $\mathbb{T}_{l'}$, then the dimension of $T_{l'}D_l$ is $n-3$ (i.e., D_l is smooth of dimension $n-3$ at l'). If l' is of second type and l is contained in $\mathbb{T}_{l'}$, then the dimension of $T_{l'}D_l$ is $n-2$.*

Proof : The variety D_l is the intersection of F with the variety G_l parametrizing all lines in $\mathbb{P}^n$ which are incident to l. Therefore $T_{l'}D_l = T_{l'}G_l \cap T_{l'}F \subset T_{l'}G(2, n+1)$. Let V and V' be the vector spaces in $\mathbb{C}^{n+1}$ whose projectivizations are respectively l and l'. Then $T_{l'}G_l$ can be identified with the subvector space of $T_{l'}G(2, n+1) = Hom(V', \mathbb{C}^{n+1}/V')$ consisting of those homomorphisms f such that $f(V \cap V') \subset (V + V')/V'$ (see e.g. [10], Ex. 16.4 pages 202-203). It follows that the set of homomorphisms f such that $f(V \cap V') = 0$ is a codimension one subspace of $T_{l'}G_l$ and therefore its intersection H with $T_{l'}D_l$ has codimension one or less in $T_{l'}D_l$.

The space $T_{l'}F$ can be identified with the subvector space of $T_{l'}G(2, n+1) = Hom(V', \mathbb{C}^{n+1}/V')$ consisting of those homomorphisms f such that for any vector $v \in V' \setminus \{0\}$ mapping to a point $p \in l'$, we have $f(v) \in T'_pX/V'$ (see [10] Ex. 16.21 and 16.23 pages 209-210). If $f : V' \to \mathbb{C}^{n+1}/V'$ verifies $f(V \cap V') = 0$, then $f(V') = \mathbb{C}f(v)$ for v a general vector in V'. Hence, if $f \in H$, then $f(V') \subset \bigcap_{p \in l'} T'_pX/V'$.

If l' is of first type, then $\bigcap_{p \in l'} T'_pX$ has dimension $n-2$, hence $\bigcap_{p \in l'} T'_pX/V'$ has dimension $n-4$. So H has dimension $n-4$ and, since H has codimension one or less in $T_{l'}D_l$, we deduce that $T_{l'}D_l$ has dimension at most $n-3$ hence it has dimension equal to $n-3$ (since D_l has dimension $\geq n-3$).

If l' is of second type, then the tangent space $T_{l'}F$ can be identified with $Hom(V', \bigcap_{p \in l'} T'_pX/V')$ (because, for instance, the latter is contained in $T_{l'}F$ and the two spaces have the same dimension). If V is not contained in $\bigcap_{p \in l'} T'_pX$, then $f(V \cap V') \subset (V+V')/V'$ for $f \in Hom(V', \bigcap_{p \in l'} T'_pX/V')$ implies $f(V \cap V') = 0$. So $T_{l'}D_l = T_{l'}F \cap T_{l'}G_l$ has dimension equal to the dimension of $\bigcap_{p \in l'} T'_pX/V'$ which is $n-3$. So in this case D_l is smooth at l'. If $V \subset \bigcap_{p \in l'} T'_pX/V'$, then the requirement $f(V \cap V') \subset (V+V')/V'$ imposes $n-4$ conditions on f and the dimension of $T_{l'}D_l$ is $n-2$. □

Since $\mathbb{T}_l$ has dimension $n-3$, we see that, as soon as $n \geq 5$, we have $l \in D_l$. We have the following

Lemma 1.3. *If $n \geq 6$, then D_l is singular at l. If $n = 5$, then D_l is smooth at l if X does not have contact multiplicity 3 along l with the plane $\mathbb{T}_l$ and if there is no line l' of second type in $\mathbb{T}_l$.*

Proof : The case $n = 5$ is Lemme 1 on page 590 of [15]. Suppose $n \geq 6$. For l general, consider a plane section of X of the form $l + l' + l''$ such that $l \cap l'$ and $l \cap l''$ are general points on l. The set of lines through $l \cap l'$ is a divisor in D_l and meets the set of lines through $l \cap l''$ only at $l \in D_l$. So we have two divisors in D_l which meet only at a point and D_l has dimension ≥ 3. Therefore D_l is not smooth at l for l general and hence for all l. □

We now prove an existence result, namely,

Lemma 1.4. *The set of lines $l \in F$ such that l is contained in $\mathbb{T}_{l'}$ for some line $l' \in F$ of second type is a proper closed subset of F. In other words (by Lemma 1.2), for $l \in F$ general, the variety $D_l \setminus \{l\}$ is smooth of dimension $n-3$.*

Proof : Since the dimension of F is $2(n-3)$ and the dimension of the variety $F_0 \subset F$ parametrizing lines of second type is $n-3$ (see [6] page 311 Corollary 7.6), if the lemma fails, then for any line $l' \in F_0$, the dimension of the family of lines in $X \cap \mathbb{T}_{l'}$ which intersect l' is at least $n-3$.

The variety $\mathbb{T}_{l'}$ is a linear subspace of codimension 2 of $\mathbb{P}^n$. Any plane in $\mathbb{T}_{l'}$ which contains l' is tangent to X along l'. The intersection of a general such plane P with X is the union of l' and a line l, the line l' having multiplicity 2 (or 3 if $l = l'$) in the intersection cycle $[P \cap X]$. Conversely, any line l in $X \cap \mathbb{T}_{l'}$ which intersects l' is contained in such a plane. The family of planes in $\mathbb{T}_{l'}$ which contain l' has dimension $n-4$. Therefore, if the family of lines l in $X \cap \mathbb{T}_{l'}$ which intersect l' has dimension $\geq n-3$, then for each such line $l \neq l'$, the plane $\langle l, l' \rangle$ contains a positive-dimensional family of lines in $X \cap \mathbb{T}_{l'}$ and hence $\langle l, l' \rangle$ is contained in $X \cap \mathbb{T}_{l'}$. Therefore $X \cap \mathbb{T}_{l'}$ is a cone over a cubic hypersurface in $\mathbb{T}_{l'}/l'$ and, for each plane $P \subset X \cap \mathbb{T}_{l'}$ which contains l', there is a hyperplane in $\mathbb{T}_{l'}$ tangent to $X \cap \mathbb{T}_{l'}$ along P. Therefore $\mathbb{T}_P \overset{def}{=} \cap_{p \in P} \mathbb{P}T'_p X$ has codimension 3 in $\mathbb{P}^n$. Hence the restriction of the dual morphism of X to P is a morphism of degree 4 from P onto a plane in $(\mathbb{P}^n)^*$. It follows from [6] Lemma 5.15 page 304 that all such planes are contained in a proper closed subset of X. Therefore a general line $l \in F$ is not contained in such a plane and hence not in $\mathbb{T}_{l'}$. Contradiction. $\square$

2. DESINGULARIZING D_l

Let X_l and $\mathbb{P}^n_l$ be the blow ups of X and $\mathbb{P}^n$ respectively along l. Then the projection from l gives a projective bundle structure on $\mathbb{P}^n_l$ and a conic bundle structure on X_l (i.e., a general fiber of $\pi_X : X_l \to \mathbb{P}^{n-2}$ is a conic in the corresponding fiber of $\pi : \mathbb{P}^n_l \to \mathbb{P}^{n-2}$):

$$\begin{tikzcd} X_l \arrow[r, hook] \arrow[dr, "\pi_X"'] & \mathbb{P}^n_l \arrow[d, "\pi"] \\ & \mathbb{P}^{n-2} \end{tikzcd}$$

Let E be the locally free sheaf $\mathcal{O}_{\mathbb{P}^{n-2}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}^{\oplus 2}$. Then it is easily seen (as in e.g. [11] page 374 Example 2.11.4) that $\pi : \mathbb{P}^n_l \to \mathbb{P}^{n-2}$ is isomorphic to the projective bundle $\mathbb{P}(E) \to \mathbb{P}^{n-2}$. The variety $X_l \subset \mathbb{P}^n_l$ is the divisor of zeros of a section s of $\mathcal{O}_{\mathbb{P}E}(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^{n-2}}(m)$ for some integer m because the general fibers of $\pi_X : X_l \to \mathbb{P}^{n-2}$ are smooth conics in the fibers of π. Since $\pi_*(\mathcal{O}_{\mathbb{P}E}(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^{n-2}}(m)) \cong Sym^2 E^* \otimes \mathcal{O}_{\mathbb{P}^{n-2}}(m)$, the section s defines a ("symmetric") morphism of vector bundles $\phi : E \to E^* \otimes \mathcal{O}_{\mathbb{P}^{n-2}}(m)$. The degeneracy locus $Q_l \subset \mathbb{P}^{n-2}$ of this morphism is the locus over which the fibers of π_X are singular conics (or have dimension ≥ 2). By, for instance, intersecting Q_l with a general line, we see that Q_l is a quintic hypersurface (see [12] pages 3-5). Therefore $m = 1$. Let S_l be the variety parametrizing lines in the fibers of π_X. We have a morphism $S_l \to D_l$ defined by sending a line in a fiber of π to its image in $\mathbb{P}^n$. Let $E_1 \subset X_l$ be the exceptional divisor of $\epsilon_1 : X_l \to X$ and let $P_1 \subset S_l$ be the variety parametrizing lines which lie in E_1. Then the morphism $S_l \to D_l$ induces an isomorphism $S_l \setminus P_1 \cong D_l \setminus \{l\}$. We have

Lemma 2.1. *Suppose that l is of first type and $D_l \setminus \{l\}$ is smooth. Then S_l is smooth and irreducible and admits a* morphism *of generic degree 2 onto Q_l. The variety S_l can also be defined as the closure of the subvariety of $G(2, n+1) \times G(3, n+1)$ parametrizing pairs (l', L') such that $l' \in D_l \setminus \{l\}$ and $L' = \langle l, l' \rangle$.*

Proof : The morphism $S_l \to Q_l$ is defined by sending a line in a fiber of π to its image in $\mathbb{P}^{n-2}$. It is of generic degree 2 because the rational map $D_l \to Q_l$ is of generic degree 2. The variety S_l is irreducible because Q_l is irreducible and $S_l \to Q_l$ is not split (intersect Q_l with a general plane and use [2]).

For $l' \in S_l \setminus P_1$, the variety S_l is smooth at l' since $S_l \setminus P_1 \cong D_l \setminus \{l\}$.

For $l' \in P_1$ we determine the Zariski tangent space to S_l at l'. Since l' maps to a point in $\mathbb{P}^{n-2}$, it corresponds to a plane L' in $\mathbb{P}^n$ which contains l. Since l' is also contained in E_1, it maps onto l in $\mathbb{P}^n$ under the blow up morphism $\mathbb{P}^n_l \to \mathbb{P}^n$ and L' is tangent to X along l. So we easily see that we can identify S_l with the closure of the subvariety of the product of the Grassmannians $G(2, n+1) \times G(3, n+1)$ parametrizing pairs (l', L') such that $l' \in D_l \setminus \{l\}$ and $L' = \langle l, l' \rangle$.

Let W' and V be the vector spaces in $\mathbb{C}^{n+1}$ whose projectivizations are L' and l respectively. The tangent space to $G(2, n+1) \times G(3, n+1)$ at (l, L') can be canonically identified with $Hom(V, \mathbb{C}^{n+1}/V) \oplus Hom(W', \mathbb{C}^{n+1}/W')$. As in [10] Ex. 16.3 pages 202-203 and Ex. 16.21, 16.23 pages 209-210, one can see that the tangent space to S_l at (l, L') can be identified with the set of pairs of homomorphisms (f, g) such that for every nonzero vector $v \in V$ mapping to a point p of l, we have $f(v) \in T'_pX/V$, $g(V) = 0$, $g|_V = f(\mathrm{mod} W')$ and $g(W') \subset \cap_{p \in l} T'_pX/W'$ (this last condition expresses the fact that the deformation of L' contains a deformation of l which is contained in X, hence the deformation of L' is tangent to X along l, i.e., is contained in $\mathbb{T}_l$). Equivalently, $g(V) = 0$, $f(V) \subset W'/V$ and $g(W') \subset \cap_{p \in l} T'_pX/W'$. Assuming l of first type, we see that the space of such pairs of homomorphisms has dimension $n - 3$. $\square$

3. The planes in X

Let $\mathcal{P}$ be the variety parametrizing planes in X. For $P \in \mathcal{P}$, we say that δ has rank r_P on P if the span of $\delta(P)$ has dimension r_P. Since δ is defined by quadrics, we have $r_P \leq 5$. Since X is smooth, we have $r_P \geq 2$. Consider the commutative diagram

$$\begin{tikzcd} & \mathbb{P}^5 \arrow[d, "p"] \\ P \arrow[ur, "v"] \arrow[r, "\delta_P"] & \mathbb{P}^{r_P} \subset (\mathbb{P}^n)^* \end{tikzcd}$$

where v is the Veronese map, δ_P is the restriction of δ to P and p is the projection from a linear space $L \subset \mathbb{P}^5$ of dimension $4 - r_P$ (with the convention that the empty set has dimension -1).

Note that L does not intersect $v(P)$ because δ is a morphism.

Let $\mathcal{P}_r$ be the subvariety of $\mathcal{P}$ parametrizing planes P for which $r_P \leq r$. In this section we will prove a few facts about $\mathcal{P}$ and $\mathcal{P}_r$ which we will need later. We begin with

Lemma 3.1. *Let* $T \overset{def}{=} \cup_{l \subset P} \langle v(l) \rangle \subset \mathbb{P}^5$ *be the secant variety of* $v(P)$. *Then there is a bijective morphism from* $T \cap L$ *to the parameter space of the family of lines of second type in* P *and* $T \cap L$ *contains no positive-dimensional linear spaces. In particular,*

1. *if* $r_P = 5$, *then* P *contains no lines of second type,*
2. *if* $r_P = 4$, *then* P *contains at most one line of second type and this happens exactly when* L *(which is a point in this case) is in* T,
3. *if* $r_P = 3$, *then* P *contains one, two or three distinct lines of second type,*
4. *if* $r_P = 2$, *then* P *contains exactly a one-parameter family of lines of second type whose parameter space is the bijective image of an irreducible and reduced plane cubic.*

Proof : A line $l \subset P$ is of second type if and only if $\delta_P(l) \subset \mathbb{P}^{r_P}$ is a line, i.e., if and only if the span $\langle v(l) \rangle \cong \mathbb{P}^2$ of the smooth conic $v(l)$ intersects L. Consider the universal line $f_1 : L_P \to P^*$ and its embedding $L_P \hookrightarrow V_P$ where $f_2 : V_P \to P^*$ is the projectivization of the bundle $f_* \mathcal{O}_{L_P}(2)^*$. Then T is the image of V_P in $\mathbb{P}^5$ by a morphism, say g, which is an isomorphism on the complement of L_P and contracts L_P onto $v(P)$. Since $L \cap v(P) = \emptyset$, the morphism $g|_{g^{-1}(T \cap L)}$ is an isomorphism, say g'. The morphism from $T \cap L$ onto the parameter space of the family of lines of second type in P is the composition of ${g'}^{-1}$ with f_2. This morphism is bijective because (since $L \cap v(P) = \emptyset$) the space L intersects any $\langle v(l) \rangle$ in at most one point and any two planes $\langle v(l_1) \rangle$ and $\langle v(l_2) \rangle$ intersect in exactly one point which is $v(l_1 \cap l_2) \in v(P)$.

To show that $T \cap L$ contains no positive-dimensional linear spaces, recall that T is the image of the Segre embedding of $P \times P$ in $\mathbb{P}^8 = \mathbb{P}\left(H^0(P, \mathcal{O}_P(1))^{\otimes 2}\right)^*$ by the projection from $\mathbb{P}\left(\Lambda^2 H^0(P, \mathcal{O}_P(1))\right)^*$. Let R_1 be the ruling of T by planes which are images of the fibers of the two projections of $P \times P$ onto P. Let R_2 be the ruling of T by planes of the form $\langle v(l) \rangle$ for some line $l \subset P$. Then a simple computation (determining all the pencils of conics which consist entirely of singular conics) shows that every linear subspace contained in T is contained either in an element of R_1 or an element of R_2. Therefore, if $L \cap T$ contains a linear space m, then either $m \subset \langle v(l) \rangle$ for some line $l \subset P$ or $m \subset L'$ for some element L' of R_1. In the first case, the space m is a point because otherwise it would intersect $v(P)$. In the second case, the space m is either a point or a line because any element of R_1 contains exactly one point of $v(P)$. It is easily seen that there is an element $s_0 \in H^0(P, \mathcal{O}_P(1))$ such that L' parametrizes the hyperplanes in $|\mathcal{O}_P(2)|$ containing all the conics of the form $Z(s.s_0)$ for some $s \in H^0(P, \mathcal{O}_P(1))$. If $m \subset L'$ is a line, then it is easily seen that the codimension, in $\langle \partial_0 G, ..., \partial_n G \rangle |_P$, of the set of elements of the form $s.s_0$ is one. Restricting to $Z(s_0)$, we see that the dimension of $\langle \partial_0 G, ..., \partial_n G \rangle |_{Z(s_0)}$ is 1 which is impossible since then X would have a singular point on $Z(s_0)$. Therefore m is always a point if non-empty. □

Proposition 3.2. *The space of infinitesimal deformations of P in X has dimension $3n-15$ if $r_P = 2$. In particular, if $n = 5$, then X contains at most a finite number of planes.*

Proof : The intersection $\mathbb{T}_P$ of the projective tangent spaces to X along P has dimension $n-3$. It follows that we have an exact sequence

$$0 \longrightarrow \mathcal{O}_P(1)^{n-5} \longrightarrow N_{P/X} \longrightarrow V_2 \longrightarrow 0$$

where V_2 is a locally free sheaf of rank 2. We need to show that $h^0(P, V_2) = 0$. Suppose that there is a nonzero section $u \in H^0(P, V_2)$. We will first show that the restriction of u to any line l in P is nonzero. This will follow if we show that the restriction map $H^0(P, V_2) \to H^0(l, V_2|_l)$ is injective, i.e., $h^0(P, V_2(-1)) = 0$. Consider therefore the exact sequence of normal sheaves

$$0 \longrightarrow N_{P/X} \longrightarrow N_{P/\mathbb{P}^n} \longrightarrow N_{X/\mathbb{P}^n}|_P \longrightarrow 0$$

After tensoring by $\mathcal{O}_P(-1)$ we obtain the exact sequence

$$0 \longrightarrow N_{P/X}(-1) \longrightarrow \mathcal{O}_P{}^{\oplus(n-2)} \longrightarrow \mathcal{O}_P(2) \longrightarrow 0.$$

We can choose our system of coordinates (on $\mathbb{P}^n$) in such a way that $x_3 = ... = x_n = 0$ are the equations for P and the map $\mathcal{O}_P{}^{\oplus(n-2)} \to \mathcal{O}_P(2)$ in the sequence above is given by multiplication by $\partial_3 G|_P, ... \partial_n G|_P$. So we see that, since $r_P = 2$, the map on global sections $H^0(\mathcal{O}_P{}^{\oplus(n-2)}) \to H^0(\mathcal{O}_P(2))$ has rank 3. Therefore $h^0(P, N_{P/X}(-1)) = n-5$ and $h^0(P, V_2(-1)) = 0$.

By Lemma 3.1, the plane P contains lines of first type. For any line $l \subset P$ which is of first type, it is easily seen that $V_2|_l \cong \mathcal{O}_l^{\oplus 2}$. Hence u has no zeros on l. It follows that $Z(u)$ is finite.

We compute the total Chern class of V_2 as

$$c(V_2) = \frac{c(N_{P/X})}{(1+\zeta)^{n-5}} = 1 + 3\zeta^2$$

where $\zeta = c_1(\mathcal{O}_P(1))$. Therefore $Z(u)$ is a finite subscheme of length 3 of P. Let l_u be a line in P such that $l_u \cap Z(u)$ has length ≥ 2. Then, by what we saw above, l_u is of second type. It is easily seen that $V_2|_{l_u} \cong \mathcal{O}_{l_u}(-1) \oplus \mathcal{O}_{l_u}(1)$. Restricting u to l_u, we see that $Z(u|_{l_u}) = l_u \cap Z(u)$ has length 1 which is a contradiction. So $h^0(P, V_2) = 0$ and $h^0(P, N_{P/X}) = 3n - 15$. □

The next result we will need is

Lemma 3.3. *The dimension of $\mathcal{P}_2$ is at most $Min(n-4, 5)$.*

Proof : The proof of the part $dim(\mathcal{P}_2) \leq n-4$ is similar to the proof of Corollary 7.6 on page 311 of [6].

To prove $dim(\mathcal{P}_2) \leq 5$, we may suppose that $n \geq 10$. Let P be an element of $\mathcal{P}_2$. We will show that the space of infinitesimal deformations of P for which the rank of δ does not increase has dimension at most 5. Let x_0, x_1, x_2 be coordinates on P, let $x_0, x_1, x_2, x_3, ..., x_{n-3}$ be coordinates on $\mathbb{T}_P$ and $x_0, ..., x_{n-3}, x_{n-2}, x_{n-1}, x_n$ coordinates on $\mathbb{P}^n$. Then the conditions $P \subset X$ and $\mathbb{T}_P$ is tangent to X along P can be written

$$\partial_i \partial_j \partial_k G = 0$$

for all $i, j \in \{0,1,2\}$, $k \in \{0, ..., n-3\}$, where G is, as before, an equation for X and $\partial_i = \frac{\partial}{\partial x_i}$. We need to determine the infinitesimal deformations of P for which there is an infinitesimal deformation of $\mathbb{T}_P$ which is tangent to X along the deformation of P. The infinitesimal deformations of P in $\mathbb{P}^n$ are parametrized by

$$Hom_{\mathbb{C}}\left(\langle \partial_0, \partial_1, \partial_2 \rangle, \frac{\mathbb{C}^{n+1}}{\langle \partial_0, \partial_1, \partial_2 \rangle}\right) \cong Hom_{\mathbb{C}}\left(\langle \partial_0, \partial_1, \partial_2 \rangle, \langle \partial_3, ..., \partial_n \rangle\right)$$

and those of $\mathbb{T}_P$ in $\mathbb{P}^n$ are parametrized by

$$Hom_{\mathbb{C}}\left(\langle \partial_0, ..., \partial_{n-3} \rangle, \frac{\mathbb{C}^{n+1}}{\langle \partial_0, ..., \partial_{n-3} \rangle}\right) \cong Hom_{\mathbb{C}}\left(\langle \partial_0, ..., \partial_{n-3} \rangle, \langle \partial_{n-2}, \partial_{n-1}, \partial_n \rangle\right)$$

where we also denote by ∂_i the vector in $\mathbb{C}^{n+1}$ corresponding to the differential operator ∂_i. We need to determine the homomorphisms $\{\partial_i \mapsto \partial'_i \in \langle \partial_3, ..., \partial_n \rangle : i \in \{0,1,2\}\}$ for which there is a homomorphism $\{\partial_i \mapsto \partial''_i \in \langle \partial_{n-2}, \partial_{n-1}, \partial_n \rangle : i \in \{0, ..., n-3\}\}$ such that the following conditions hold.

1. The vector ∂''_i is the projection of ∂'_i to $\langle \partial_{n-2}, \partial_{n-1}, \partial_n \rangle$ for $i \in \{0,1,2\}$. This expresses the condition that the infinitesimal deformation of $\mathbb{T}_P$ contains the infinitesimal deformation of P.
2. For all $i, j \in \{0,1,2\}$ and $k \in \{0, ..., n-3\}$,

$$(\partial_i + \epsilon\partial'_i)\left(\partial_j + \epsilon\partial'_j\right)(\partial_k + \epsilon\partial''_k)\, G = 0$$

where, as usual, ϵ has square 0. Here we are "differentiating" the relations $\partial_i\partial_j\partial_k G = 0$. Developing, we obtain

$$\left(\partial_i\partial_j\partial''_k + \partial_i\partial'_j\partial_k + \partial'_i\partial_j\partial_k\right) G = 0 \ .$$

Writing $\partial'_i = \sum_{j=3}^{n} a_{ij}\partial_j$ and $\partial''_i = \sum_{j=n-2}^{n} b_{ij}\partial_j$, the above conditions can be written as

1. For all $i \in \{0,1,2\}$ and $j \in \{n-2, n-1, n\}$,

$$a_{ij} = b_{ij} \ .$$

2. For all $i, j \in \{0,1,2\}$ and $k \in \{0, ..., n-3\}$,

$$\sum_{l=n-2}^{n} b_{kl}\partial_i\partial_j\partial_l G + \sum_{l=3}^{n} a_{jl}\partial_i\partial_l\partial_k G + \sum_{l=3}^{n} a_{il}\partial_l\partial_j\partial_k G = 0 \ .$$

Incorporating the first set of conditions in the second and using the relations $\partial_i\partial_j\partial_k G = 0$ for $i, j \in \{0,1,2\}, k \in \{0, ... n-3\}$, we divide our conditions into two different sets of conditions as follows. We are looking for matrices $(a_{il})_{0 \le i \le 2, 3 \le l \le n}$ for which there is a matrix $(b_{kl})_{3 \le k \le n-3, n-2 \le l \le n}$ such that, for all $i, j, k \in \{0,1,2\}$,

$$\sum_{l=n-2}^{n} \left(a_{kl}\partial_i\partial_j\partial_l + a_{jl}\partial_i\partial_l\partial_k + a_{il}\partial_l\partial_j\partial_k\right) G = 0$$

and, for all $i, j \in \{0,1,2\}$, $k \in \{3, ..., n-3\}$,

$$\sum_{l=n-2}^{n} b_{kl}\partial_i\partial_j\partial_l G + \sum_{l=3}^{n} \left(a_{jl}\partial_i\partial_l\partial_k + a_{il}\partial_l\partial_j\partial_k\right) G = 0 \ .$$

Consider the matrix whose columns are indexed by the a_{lm}, b_{su} ($0 \le l \le 2$, $3 \le m \le n$, $3 \le s \le n-3$, $n-2 \le u \le n$), whose rows are indexed by *unordered* triples (i, j, k) with $i, j \in \{0,1,2\}$, $k \in \{0, ..., n-3\}$ and whose entries are the $\partial_i\partial_j\partial_m G$, $\partial_i\partial_m\partial_k G$, $\partial_m\partial_j\partial_k G$ or $\partial_i\partial_j\partial_u G$. The entry in the column of a_{lm} and the row of (i, j, k) is nonzero only if $l = i, j$ or k. We can, and will, suppose that $l = i$. Here is the list of possibly nonzero such entries.

$$3 \le m \le n \ , \quad 3 \le k \le n-3$$

$l = i \neq j \quad \partial_m\partial_j\partial_k G$

$l = i = j \quad 2\partial_m\partial_l\partial_k G$

$$n-2 \le m \le n \ , \quad 0 \le k \le 2$$

$$\begin{array}{ll} l = i \neq j,k & \partial_m \partial_j \partial_k G \\ l = i = j \neq k & 2\partial_m \partial_l \partial_k G \\ l = i = j = k & 3\partial_m \partial_l^2 G \ . \end{array}$$

The entry in the column of b_{su} and the row of $\{i,j,k\}$ is nonzero only if $s = k$. These possibly nonzero entries are the following.

$$n-2 \leq u \leq n \ , \ \ 3 \leq k \leq n-3$$

$$s = k \quad \partial_i \partial_j \partial_u G \ .$$

An easy dimension count shows that we need to prove that there are at most 6 relations between the rows of the matrix. Suppose that there are t relations with coefficients

$$\{\{\lambda^r_{ijk}\}_{\substack{0 \leq i,j \leq 2 \\ 0 \leq k \leq n-3}}\}_{1 \leq r \leq t}$$

between the rows of our matrix. Each relation can be written as a collection

$3 \leq m \leq n-3$, $0 \leq i \leq 2$

$$\sum_{\substack{3 \leq k \leq n-3 \\ 0 \leq j \leq 2}} \lambda^r_{ijk} \partial_m \partial_j \partial_k G = 0$$

$n-2 \leq m \leq n$, $0 \leq i \leq 2$

$$\sum_{\substack{0 \leq k \leq n-3 \\ 0 \leq j \leq 2}} \lambda^r_{ijk} \partial_m \partial_j \partial_k G = 0 \tag{1}$$

$n-2 \leq u \leq n$, $3 \leq k \leq n-3$

$$\sum_{0 \leq i,j \leq 2} \lambda^r_{ijk} \partial_i \partial_j \partial_u G = 0$$

Each expression $\sum_{0 \leq i,j \leq 2} \lambda^r_{ijk} \partial_i \partial_j$ defines a hyperplane in $H^0(P, \mathcal{O}_P(2))$ which contains the polynomials $\partial_u G|_P$. Since we have 3 independant such polynomials, the vector space of hyperplanes containing them has dimension 3. Hence, after a linear change of coordinates, we can suppose that, for $r \in \{0, ..., t-3\}$, we have $\lambda^r_{ijk} = 0$ if $0 \leq i,j \leq 2, 3 \leq k \leq n-3$. The relations (1) now become

$0 \leq r \leq t-3$, $0 \leq i \leq 2$

$$\sum_{\substack{0 \leq k \leq 2 \\ 0 \leq j \leq 2}} \lambda^r_{ijk} \partial_j \partial_k G = 0 \ .$$

If, for a fixed $r \in \{1, ..., t-3\}$, the three relations $\sum_{\substack{0 \leq k \leq 2 \\ 0 \leq j \leq 2}} \lambda^r_{ijk} \partial_j \partial_k G = 0$, for $0 \leq i \leq 2$, are not independent, then after a linear change of coordinates, we may suppose that, for instance, $\lambda^r_{2jk} = 0$ for all $j,k \in \{0,1,2\}$. Since the coefficients λ^r_{ijk} are symmetric in i,j,k, we obtain

$0 \leq i \leq 1$

$$\sum_{\substack{0\leq k\leq 1\\ 0\leq j\leq 1}} \lambda^r_{ijk}\partial_j\partial_k G = 0\ .$$

If l is the line in P obtained as the projectivization of $\langle \partial_0, \partial_1\rangle$, then $\langle \partial_{n-2}G, \partial_{n-1}G, \partial_n G\rangle|_l$ has dimension at least 2 and there can be at most one hyperplane in $H^0(l, \mathcal{O}_l(2))$ containing $\langle \partial_{n-2}G, \partial_{n-1}G, \partial_n G\rangle|_l$. In other words, up to multiplication by a scalar, there is at most one nonzero relation $\sum_{\substack{0\leq k\leq 1\\ 0\leq j\leq 1}} \lambda^r_{ijk}\partial_j\partial_k G = 0$. Hence, we can suppose $\lambda^r_{1jk} = 0$ for all $j, k \in \{0, 1\}$. Again, by symmetry, we are reduced to $\lambda^r_{000}\partial_0^2 G = 0$ which implies $\lambda^r_{000} = 0$ because X is smooth. Hence all the λ^r_{ijk} are zero.

Therefore, if the λ^r_{ijk} are not all zero, the three relations

$0 \leq i \leq 2$

$$\sum_{\substack{0\leq k\leq 2\\ 0\leq j\leq 2}} \lambda^r_{ijk}\partial_j\partial_k G = 0$$

are independent. If $t-3 \geq 4$, then, after a linear change of coordinates, for some $r \in \{1, ..., t-3\}$, one of the above three relations will be trivial and we are reduced to the previous case. Therefore $t-3 \leq 3$ and $t \leq 6$. $\square$

Proposition 3.4. *Suppose that $n \geq 6$. Then $\mathcal{P}$ has pure dimension equal to the expected dimension $3n-16$. If $r_P \geq 3$, then $\mathcal{P}$ is smooth at P.*

Proof : Since the dimension of $\mathcal{P}_2$ is at most $Min(n-4, 5)$ by Lemma 3.3 and the dimension of every irreducible component of $\mathcal{P}$ is at least $3n-16$, it is enough to show that for every P such that $r_P \geq 3$, the space $H^0(P, N_P)$ of infinitesimal deformations of P in X has dimension $3n-16$.

Suppose that $r_P = 3$. As in the proof of Proposition 3.2, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_P(1)^{\oplus(n-6)} \longrightarrow N_{P/X} \longrightarrow V_3 \longrightarrow 0$$

where V_3 is a locally free sheaf of rank 3. Since $h^0(P, N_{P/X}) \geq 3n-16$, we have $h^0(P, V_3) \geq 2$. We need to show that $h^0(P, V_3) = 2$. As in the proof of Proposition 3.2 we have $h^0(P, V_3(-1)) = 0$ so that, for any line $l \subset P$,

$$H^0(P, V_3) \hookrightarrow H^0(l, V_3|_l)\ .$$

Suppose that $h^0(P, V_3) \geq 3$ and let u_1, u_2, u_3 be three linearly independent elements of $H^0(P, V_3)$. By Lemma 3.1, the plane P contains at least one line l_0 of second type. It is easily seen that $V_3|_{l_0} \cong \mathcal{O}_{l_0}(-1) \oplus \mathcal{O}_{l_0}(1)^{\oplus 2}$. Therefore $\langle u_1, u_2, u_3\rangle|_{l_0}$ generates a subsheaf of the $\mathcal{O}_{l_0}(1)^{\oplus 2}$ summand of $V_3|_{l_0}$ isomorphic to $\mathcal{O}_{l_0} \oplus \mathcal{O}_{l_0}(1)$. The quotient of $\mathcal{O}_{l_0}(1)^{\oplus 2}$ by $\mathcal{O}_{l_0} \oplus \mathcal{O}_{l_0}(1)$ is a skyscraper sheaf supported on a point p of l_0 (with fiber at p isomorphic to $\mathbb{C}$). So the images of u_1, u_2 and u_3 by the evaluation map at p generate a one-dimensional vector subspace of the fiber of V_3 at p. By Lemma 3.1, there is a line l of first type in P which contains p. It is easily seen that $V_3|_l \cong \mathcal{O}_l^{\oplus 2} \oplus \mathcal{O}_l(1)$. Restricting u_1, u_2, u_3 to l we see that their images by the evaluation map at p generate a vector subspace of dimension ≥ 2 of the fiber of V_3 at p. Contradiction.

Suppose now that $r_P = 4$. Then $n \geq 7$ and we have the exact sequence

$$0 \longrightarrow \mathcal{O}_P(1)^{\oplus(n-7)} \longrightarrow N_{P/X} \longrightarrow V_4 \longrightarrow 0$$

where V_4 is a locally free sheaf of rank 4. Since $h^0(P, N_{P/X}) \geq 3n - 16$, we have $h^0(P, V_4) \geq 5$. We need to show that $h^0(P, V_4) = 5$. As before, $h^0(P, V_4(-1)) = 0$, hence, for any line $l \subset P$, we have $H^0(P, V_4) \hookrightarrow H^0(l, V_4|_l)$. It is easily seen that when l is of first type $V_4|_l \cong \mathcal{O}_l^{\oplus 2} \oplus \mathcal{O}_l(1)^{\oplus 2}$ and when l is of second type $V_4|_l \cong \mathcal{O}_l(-1) \oplus \mathcal{O}_l(1)^{\oplus 3}$. Thus $h^0(P, V_4) \leq 6$. Suppose that $h^0(P, V_4) = 6$. Then $H^0(P, V_4) \overset{\cong}{\to} H^0(l, V_4|_l)$ for every line $l \subset P$.

Suppose first that P contains a line l_0 of second type and let l be a line of first type in P. We see that V_4 is not generated by its global sections anywhere on l_0 whereas $V_4|_l$ is generated by its global sections. This gives a contradiction at the point of intersection of l and l_0.

So every line l in P is of first type, $V_4|_l \cong \mathcal{O}_l^{\oplus 2} \oplus \mathcal{O}_l(1)^{\oplus 2}$ and V_4 is generated by its global sections. Let s be a general global section of V_4. We claim that s does not vanish at any point of P. Indeed, since V_4 is generated by its global sections, for every point p of P, the vector space of global sections of V_4 vanishing at p has dimension 2. Hence the set of all global sections of V_4 vanishing at some point of P has dimension $\leq 2 + 2 = 4 < 6$. So we have the exact sequence

$$0 \longrightarrow \mathcal{O}_P \xrightarrow{s} V_4 \longrightarrow V \longrightarrow 0$$

where V is a locally free sheaf of rank 3. Since V_4 is generated by its global sections, so is V and we have $h^0(P, V) = 5$. As before a general global section s' of V does not vanish anywhere on P and we have the exact sequence

$$0 \longrightarrow \mathcal{O}_P \xrightarrow{s'} V \longrightarrow V' \longrightarrow 0$$

where V' is a locally free sheaf of rank 2. We have $h^0(P, V') = 4$ and $h^0(V'(-1)) = h^0(V(-1)) = h^0(V_4(-1)) = 0$. Hence for every line $l \subset P$, $H^0(P, V') \hookrightarrow H^0(l, V'|_l)$. Since $V'|_l \cong \mathcal{O}_l(1)^{\oplus 2}$, for a nonzero section s of V' the scheme $Z(s|_l) = Z(s) \cap l$ has length ≤ 1. The scheme $Z(s)$ is not a line because $H^0(P, V') \to H^0(Z(s), V'|_{Z(s)})$ is injective. Hence for a general line $l \subset P$, $Z(s) \cap l$ is empty.

Therefore $Z(s)$ is finite. We compute $c(V') = c(V) = c(V_4) = 1 + 2\zeta + 4\zeta^2$. Therefore $Z(s)$ has length 4. Hence there is a line l such that $Z(s_l)$ has length ≥ 2 and this contradicts $length(Z(s_l)) \leq 1$.

If $r_P = 5$, consider again the exact sequence of normal sheaves

$$0 \longrightarrow N_{P/X} \longrightarrow N_{P/\mathbb{P}^n} \longrightarrow N_{X/\mathbb{P}^n}|_P \longrightarrow 0$$

which after tensoring by $\mathcal{O}_P(-1)$ becomes

$$0 \longrightarrow N_{P/X}(-1) \longrightarrow \mathcal{O}_P^{\oplus(n-2)} \longrightarrow \mathcal{O}_P(2) \longrightarrow 0\,.$$

Then the map on glabal sections

$$H^0(P, \mathcal{O}_P^{\oplus(n-2)}) \longrightarrow H^0(P, \mathcal{O}_P(2))$$

is onto (see the proof of Proposition 3.2). A fortiori, the map

$$H^0(P, N_{P/\mathbb{P}^n}) = H^0(P, \mathcal{O}_P(1)^{\oplus(n-2)}) =$$

$$= H^0(P, \mathcal{O}_P^{\oplus(n-2)}) \otimes H^0(P, \mathcal{O}_P(1)) \longrightarrow H^0(P, \mathcal{O}_P(3)) = H^0(P, N_{X/\mathbb{P}^n}|_P)$$

is onto and $H^0(P, N_{P/X})$ has dimension $3n - 16$. ☐

Corollary 3.5. *If* $n \geq 8$, *then* $\mathcal{P}$ *is irreducible.*

Proof : As before, let G be an equation for X. Choose a linear embedding $\mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$. Choose coordinates $\{x_0, ..., x_n\}$ on $\mathbb{P}^n$ and coordinates $\{x_0, ..., x_n, x_{n+1}\}$ on $\mathbb{P}^{n+1}$. Let $Y \subset \mathbb{P}^{n+1}$ be the cubic of equation $G + x_{n+1}Q$ where Q is the equation of a general quadric in $\mathbb{P}^{n+1}$ and let $\mathcal{P}_Y \supset \mathcal{P}$ be the variety of planes in Y. Then, by Proposition 3.4, the codimension of $\mathcal{P}$ in $\mathcal{P}_Y$ is 3. The singular locus of $\mathcal{P}$ is $\mathcal{P}_2$ (3.2 and 3.4) which has codimension at least 4 in $\mathcal{P}$ by 3.3 and 3.4. Therefore, since $\mathcal{P}$ is connected ([5] Theorem 4.1 page 33 or [7] Théorème 2.1), it is sufficient to show that $\mathcal{P}_Y$ is smooth at a general point of $\mathcal{P}_2$. Since Q does not contain a general plane $P \in \mathcal{P}_2$, the rank of the dual morphism of Y on P is at least 3. Hence $\mathcal{P}_Y$ is smooth at a general point of $\mathcal{P}_2$ (3.4). ☐

Lemma 3.6. *The dimension of* $\mathcal{P}_3$ *is at most* $n - 2$.

Proof : It is enough to show that at any P with $r_P \leq 3$ the dimension of the tangent space to $\mathcal{P}_3$ is at most $n - 2$. By 3.3 it is enough to prove this for $r_P = 3$. The proof of this is very similar to (and simpler than) the proof of Lemma 3.3. ☐

Proposition 3.7. *If* $n \geq 7$, *then* $\mathcal{P}_4$ *has pure dimension* $2n - 9$.

Proof : For $n = 7$ there is nothing to prove since $\mathcal{P}$ has pure dimension $5 = 3.7 - 16 = 2.7 - 9$ and $\mathcal{P} = \mathcal{P}_4$.

Suppose $n \geq 8$. By an easy dimension count, the dimension of every irreducible component of $\mathcal{P}_4$ is at least $2n - 9$. Since the dimension of $\mathcal{P}_3$ is at most $n - 2 < 2n - 9$ (see 3.6), for a general element P of any irreducible component of $\mathcal{P}_4$ we have $r_P = 4$. We first show

Lemma 3.8. *Suppose $n \geq 8$. Then the subscheme $\mathcal{P}'_4$ of $\mathcal{P}_4$ parametrizing planes which contain a line of second type has pure dimension $2n - 10$.*

Proof : Again by a dimension count, the dimension of every irreducible component of $\mathcal{P}'_4$ is at least $2n - 10$. Let P be an element of $\mathcal{P}'_4$. By 3.6, the scheme $\mathcal{P}_3 \subset \mathcal{P}'_4$ has dimension $\leq n - 2 \leq 2n - 10$, so we may suppose that $r_P = 4$. Let l be the unique line of second type contained in P (see Lemma 3.1). Since the family of lines of second type in X has dimension $n - 3$ (see [6] Corollary 7.6), it is enough to show that the space of infinitesimal deformations of P in X which contain l has dimension $n - 7$.

Consider the exact sequence of sheaves

$$0 \longrightarrow N_{P/X}(-1) \longrightarrow N_{P/X} \longrightarrow N_{P/X}|_l \longrightarrow 0$$

with associated cohomology sequence

$$0 \longrightarrow H^0(P, N_{P/X}(-1)) \longrightarrow H^0(P, N_{P/X}) \longrightarrow H^0(P, N_{P/X}|_l) \longrightarrow H^1(P, N_{P/X}(-1)) \longrightarrow ...$$

The space of infinitesimal deformations of P in X which contain l can be identified with the kernel of the homomorphism $H^0(P, N_{P/X}) \to H^0(P, N_{P/X}|_l)$ which, by the above sequence, can be identified with $H^0(P, N_{P/X}(-1))$. Recall the exact sequence

$$0 \longrightarrow N_{P/X}(-1) \longrightarrow \mathcal{O}_P^{\oplus(n-2)} \longrightarrow \mathcal{O}_P(2) \longrightarrow 0$$

where the map $\mathcal{O}_P^{\oplus(n-2)} \to \mathcal{O}_P(2)$ is given by multiplication by $\partial_3 G, ..., \partial_n G$ (see the proof of 3.2). It immediately follows that $h^0(P, N_{P/X}(-1)) = n - 7$ if and only if $r_P = 4$. ☐

Note that containing a line of second type imposes at most one condition on planes P with $r_P \leq 4$. Therefore Proposition 3.7 follows from Lemma 3.8. ☐

4. RESOLVING THE INDETERMINACIES OF THE RATIONAL INVOLUTION ON S_l

A good generalization of the Prym construction for cubic threefolds to cubic hypersurfaces of higher dimension would be to realize the cohomology of X as the anti-invariant part of the cohomology of S_l for the involution exchanging two lines whenever they are in the same fiber of π. However, this is only a rational involution and we need to resolve its indeterminacies. This involution is not well-defined exactly at the lines l' such that $\pi^{-1}(\pi(l')) \subset X_l$, i.e., the plane $L' \subset \mathbb{P}^n$ corresponding to $\pi(l')$ is contained in X. Let $T_l \subset Q_l \subset \mathbb{P}^{n-2}$ be the variety parametrizing the planes in $\mathbb{P}^n$ which contain l and are contained in X (equivalently, the variety T_l parametrizes the fibers of π which are contained in X_l). Recall that $X_l \subset \mathbb{P}^n_l$ is the divisor of zeros of $s \in H^0(\mathbb{P}E, \mathcal{O}_{\mathbb{P}E}(2) \otimes \pi^*\mathcal{O}_{\mathbb{P}^{n-2}}(1)) = H^0(\mathbb{P}^{n-2}, \pi_*(\mathcal{O}_{\mathbb{P}E}(2)) \otimes \mathcal{O}_{\mathbb{P}^{n-2}}(1)) = H^0(\mathbb{P}^{n-2}, Sym^2E^* \otimes \mathcal{O}_{\mathbb{P}^{n-2}}(1))$. Since $E \cong \mathcal{O}_{\mathbb{P}^{n-2}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}^{\oplus 2}$, we have $Sym^2E^* \otimes \mathcal{O}_{\mathbb{P}^{n-2}}(1) \cong \mathcal{O}_{\mathbb{P}^{n-2}}(3) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(1)^{\oplus 3}$. The variety T_l is the locus of common zeros of all the components of s in the above direct sum decomposition. Therefore T_l is the scheme-theoretic intersection of three hyperplanes, two quadrics and one cubic in $\mathbb{P}^{n-2}$. We have

Lemma 4.1. *There is a Zariski-dense open subset of F parametrizing lines l such that l is of first type and $r_P = 5$ for every plane P in X containing l. For l in this Zariski-dense open subset, the variety T_l is the smooth complete intersection of the six hypersurfaces obtained as the zero loci of the components of s in the direct sum decomposition $Sym^2E^* \otimes \mathcal{O}_{\mathbb{P}^{n-2}}(1) \cong \mathcal{O}_{\mathbb{P}^{n-2}}(3) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(1)^{\oplus 3}$.*

Proof : The first part of the lemma follows from Proposition 3.7. For the second part we need to show that T_l is smooth of the expected dimension $n-8$. In other words, for any plane P containing l, the space of infinitesimal deformations of P in X containing l has dimension $n-8$. The proof of this is similar to the proof of Lemma 3.8. □

Definition 4.2. *Let U_0 be the subvariety of F parametrizing lines l such that l is of first type, is not contained in $\mathbb{T}_{l'}$ for any line l' of second type and every plane containing l is an element of $\mathcal{P} \setminus \mathcal{P}_4$.*

By Lemmas 1.4 and 4.1, the variety U_0 is an open dense subvariety of F. Suppose $l \in U_0$. By Lemmas 1.2, 2.1 and 4.1, the varieties S_l and T_l are smooth of the expected dimensions $n-3$ and $n-8$ respectively. Let $X'_l \subset \mathbb{P}^{n'}_l$ be the blow ups of $X_l \subset \mathbb{P}^n_l$ along $\pi^{-1}(T_l)$ and let $\mathbb{P}^{n-2'}$ be the blow up of $\mathbb{P}^{n-2}$ along T_l. Then we have morphisms

$$\begin{tikzcd}
X'_l \arrow[r, phantom, "\subset"] \arrow[dr, "\pi'_X"'] & \mathbb{P}^{n'}_l \arrow[d, "\pi'"] \\
& \mathbb{P}^{n-2'}
\end{tikzcd}$$

where $\pi' : \mathbb{P}^{n'}_l \to \mathbb{P}^{n-2'}$ is again a $\mathbb{P}^2$-bundle. Since T_l is the zero locus of $s \in H^0(\mathbb{P}^{n-2}, \pi_*\mathcal{O}_{\mathbb{P}E}(2) \otimes \mathcal{O}_{\mathbb{P}^{n-2}}(1))$, we have $N_{T_l/\mathbb{P}^{n-2}} \cong \pi_*\mathcal{O}_{\mathbb{P}E}(2) \otimes \mathcal{O}_{\mathbb{P}^{n-2}}(1)|_{T_l}$. Therefore, the exceptional divisor E' of $\mathbb{P}^{n-2'} \to \mathbb{P}^{n-2}$ is a $\mathbb{P}^5$-bundle over T_l whose fiber at a point $t \in T_l$ corresponding to the plane $P_t \subset X_l$ is $|\mathcal{O}_{P_t}(2)|$. We have

Lemma 4.3. *Suppose that* $l \in U_0$. *For all* $t \in T_l$, *the restriction of* $\pi'_X : X'_l \to \mathbb{P}^{n-2'}$ *to* $|\mathcal{O}_{P_t}(2)| \subset \mathbb{P}^{n-2'}$ *is the universal conic on* $|\mathcal{O}_{P_t}(2)|$. *In particular, the fibers of* $\pi'_X : X'_l \to \mathbb{P}^{n-2'}$ *are always one-dimensional.*

Proof : The restriction of π' to the inverse image of a point $t \in T_l$ is the second projection $P_t \times |\mathcal{O}_{P_t}(2)| \to |\mathcal{O}_{P_t}(2)|$. Let $N_{X,p}$ be the normal space in X_l to $\pi^{-1}(T_l)$ at $p \in P_t$ and let $\rho_t : P_t \to |\mathcal{O}_{P_t}(2)|^* \cong \mathbb{P}^5$ be the map which to $p \in P_t$ associates $\mathbb{P}N_{X,p} \in |\mathcal{O}_{P_t}(2)|^*$. For $n \in |\mathcal{O}_{P_t}(2)|$, the fiber of π'_X at $(t,n) \in E'$ is equal to $\rho_t^{-1}(\rho_t(P_t) \cap H_n)$ where H_n is the hyperplane in $|\mathcal{O}_{P_t}(2)|^*$ corresponding to n. It is immediately seen that ρ_t is induced by the dual morphism δ of X. Hence, since $r_{P_t} = 5$, the map ρ_t is the Veronese morphism $P_t \to |\mathcal{O}_{P_t}(2)|^*$. Hence $\rho_t^{-1}(\rho_t(P_t) \cap H_n)$ is the conic in P_t corresponding to n. ☐

It follows from lemma 4.3 that if we let S'_l be the variety parametrizing lines in the fibers of $\pi'_X : X'_l \to \mathbb{P}^{n-2'}$, then there is a well-defined involution $i_l : S'_l \to S'_l$ which sends l' to l'' when $l' + l''$ is a fiber of $X'_l \to \mathbb{P}^{n-2'}$. Sending a line in a fiber of π'_X to its image in X_l defines a morphism $S'_l \to S_l$. Let $\mathcal{P}_l \to T_l$ be the family of planes in X containing l, then the inverse image of T_l in S_l by the morphism $S_l \to Q_l$ is the projective bundle $\mathcal{P}_l^*$ of lines in the fibers of $\mathcal{P}_l \to T_l$. We have

Proposition 4.4. *Suppose that* $l \in U_0$. *The morphism* $S'_l \to S_l$ *is the blow up of* S_l *along* $\mathcal{P}_l^*$. *In particular, the variety* S'_l *is smooth. The fixed point locus* R'_l *of* i_l *in* S'_l *is a smooth subvariety of codimension* 2 *of* S'_l. *The projective bundle* $\mathbb{P}(N_{R'_l/S'_l}) \to R'_l$ *is isomorphic to the family of lines in the fibers of* π'_X *parametrized by* R'_l.

Proof : In Lemma 2.1, we saw that S_l can be identified with the closure of the subvariety $G(2,n+1) \times G(3,n+1)$ parametrizing pairs (l', L') of a line and a plane such that $l \neq l'$ and $l \cup l' \subset L'$. In the same way, we see that S'_l can be identified with the closure of the subvariety of $G(2,n+1) \times G(2,n+1) \times G(3,n+1)$ parametrizing triples (l', l'', L') such that $L' \cap X \supset l \cup l' \cup l''$ and l, l', l'' are distinct. Furthermore, the morphism $S'_l \to S_l$ is the restriction of the projection to the second and third factors of $G(2,n+1) \times G(2,n+1) \times G(3,n+1)$. Again as in the proof of Lemma 2.1 we see that S'_l is smooth. Blowing up $\mathcal{P}_l^*$ and its inverse image in S'_l we obtain the commutative diagram

$$\begin{tikzcd}
\widetilde{S}'_l \arrow[r] \arrow[d] & \widetilde{S}_l \arrow[d] \\
S'_l \arrow[r] & S_l.
\end{tikzcd}$$

Since the inverse image of $\mathcal{P}_l^*$ is a divisor in S_l', the blow up morphism $\widetilde{S}_l' \to S_l'$ is an isomorphism. The morphism $S_l' \to \widetilde{S}_l$ thus obtained is a birational morphism of smooth varieties with constant fiber dimension hence it is an isomorphism. This proves the first part of the Proposition.

Now let Δ be the diagonal of $G(2, n+1) \times G(2, n+1)$. Then the variety R_l' is identified with $S_l' \cap (\Delta \times G(3, n+1))$. One now computes the tangent space to R_l' as in the proof of Lemma 2.1 and see that $N_{R_l'/S_l'}$ is isomorphic to $I^* \otimes J/I$ where I is the restriction of the universal bundle on $G(2, n+1)$ and J is the restriction of the universal bundle on $G(3, n+1)$. Therefore $\mathbb{P}(N_{R_l'/S_l'})$ is isomorphic to $\mathbb{P}(I)$ which is the family of lines in the fibers of π_X' parametrized by R_l'. □

Let Q_l' be the blow up of Q_l along T_l. Sending a line $l \in S_l'$ to the fiber of $X_l' \to \mathbb{P}^{n-2'}$ which contains it defines a finite morphism $S_l' \to Q_l'$ of degree 2 with ramification locus R_l'. Blowing up R_l' in Q_l' and S_l' we obtain the morphism $S_l'' \to Q_l''$. We have

Proposition 4.5. *The variety R_l' is an ordinary double locus for Q_l'. In particular, Q_l'' is smooth and (by 4.4) the projectivization $\mathbb{P}(C_{R_l'/Q_l''})$ of the normal cone to R_l' in Q_l' is isomorphic to $\mathbb{P}(N_{R_l'/S_l'})$.*

Proof : The fact that $R_l \setminus T_l$ is an ordinary double locus for $Q_l \setminus T_l$ can be proved, for instance, by intersecting Q_l with a general plane through a point p of $R_l \setminus T_l$. The resulting curve has an ordinary double point at p by [1] Proposition 1.2 page 321. At a point q of the exceptional divisor of $R_l' \to R_l$, locally trivialize the pull-back of $E = \mathcal{O}_{\mathbb{P}^{n-2}}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(-1)$ to obtain a morphism from a neighborhood U of q to $|\mathcal{O}_{\mathbb{P}^2}(2)|$. It easily follows from 4.1 and 4.3 that this morphism is dominant and the restriction of $X_l' \to \mathbb{P}^{n-2'}$ to U is the inverse image of the universal conic on $|\mathcal{O}_{\mathbb{P}^2}(2)|$. The assertion of the Proposition now follows from the corresponding fact for the cubic fourfold parametrizing singular conics in $\mathcal{P}^2$. □

5. THE MAIN THEOREM

Let $L_l \to S_l'$ and $\overline{L}_l \to S_l$ be the families of lines in the fibers of π_X' and π_X respectively. The blow-up morphism $\epsilon_2 : X_l' \to X_l$ defines a morphism $L_l \to \overline{L}_l$ which fits into the commutative diagram

$$\begin{tikzcd}
X_l' \arrow[r, "\epsilon_2"] & X_l \arrow[r, "\epsilon_1"] & X \\
L_l \arrow[u, "\rho"] \arrow[r] \arrow[d, "p"'] & \overline{L}_l \arrow[u, "\overline{\rho}"'] \arrow[d, "\overline{p}"] & \\
S_l' \arrow[r] & S_l &
\end{tikzcd}$$

where the squares are Cartesian. Put $q = \epsilon_1\epsilon_2\rho$ and let $\psi' = q_*p^* : H^{n-3}(S_l', \mathbb{Z}) \to H^{n-1}(X, \mathbb{Z})$ and $\psi = (\epsilon_1\overline{\rho})_*\overline{p}^* : H^{n-3}(S_l, \mathbb{Z}) \to H^{n-1}(X, \mathbb{Z})$ be the Abel-Jacobi maps. The map ψ is the composition of ψ' with the inclusion $H^{n-3}(S_l, \mathbb{Z}) \hookrightarrow H^{n-3}(S_l', \mathbb{Z})$ because the bottom (or top) square above is Cartesian. We have

Theorem 5.1. *The maps* $\psi : H^{n-3}(S_l, \mathbb{Z}) \to H^{n-1}(X, \mathbb{Z})$ *and* $\psi' : H^{n-3}(S_l', \mathbb{Z}) \to H^{n-1}(X, \mathbb{Z})$ *are onto.*

Proof : Consider the rational map $Q_l' \to X_l'$ which to the singular conic $l' + l''$ associates the point of intersection $l' \cap l''$. An easy local computation shows that the closure of the image of this map is smooth, hence, by a reasoning analogous to the proof of Proposition 4.4, it can be identified with Q_l''. Let $\epsilon_3 : X_l'' \to X_l'$ be the blow up of X_l' along Q_l'' and, for each i $(1 \leq i \leq 3)$, let E_i be the exceptional divisor of the blow up map ϵ_i. Then we have a factorization

$$\begin{tikzcd}
& X_l'' \arrow[d, "\epsilon_3"] \\
L_l \arrow[ur, "\tilde{q}"] \arrow[r, "\rho"] & X_l'
\end{tikzcd}$$

so that $\psi' = q_*p^* = \epsilon_{1*}\epsilon_{2*}\rho_*p^* = \epsilon_{1*}\epsilon_{2*}\epsilon_{3*}\tilde{q}_*p^*$. Note that $\tilde{q}$ is an embedding so that we can, and will, identify L_l with $\tilde{q}(L_l)$. Put $U_l = X_l'' \setminus (E_3 \cup L_l) = X_l' \setminus \rho(L_l)$. Let $m_l : U_l \to X_l''$ be the inclusion. We have the spectral sequence

$$E_2^{p,q} = H^p(X_l'', R^q m_{l*}\mathbb{Z}_{U_l}) \implies H^{p+q}(U_l, \mathbb{Z})$$

and by [8], §3.1, we have $R^0 m_{l*}\mathbb{Z}_{U_l} = \mathbb{Z}_{X_l''}$, $R^1 m_{l*}\mathbb{Z}_{U_l} = \mathbb{Z}_{E_3} \oplus \mathbb{Z}_{L_l}$, $R^2 m_{l*}\mathbb{Z}_{U_l} = \mathbb{Z}_{E_3 \cap L_l}$ and $R^q m_{l*}\mathbb{Z}_{U_l} = 0$ for $q > 2$. Note that $E_3 \cap L_l \cong S_l''$.

Therefore

$$\begin{gathered}
E_2^{p,0} = H^p(X_l'', \mathbb{Z})\,, \\
E_2^{p,1} = H^p(X_l'', \mathbb{Z}_{E_3} \oplus \mathbb{Z}_{L_l}) = H^p(L_l, \mathbb{Z}) \oplus H^p(E_3, \mathbb{Z})\,, \\
E_2^{p,2} = H^p(X_l'', \mathbb{Z}_{S_l''}) = H^p(S_l'', \mathbb{Z})\,, \\
E_2^{p,q} = 0 \text{ for } q > 2
\end{gathered}$$

So the $E_2^{\cdot\cdot}$ complex is

$$0 \longrightarrow H^{p-2}(S_l'', \mathbb{Z}) \longrightarrow H^p(L_l, \mathbb{Z}) \oplus H^p(E_3, \mathbb{Z}) \longrightarrow H^{p+2}(X_l'', \mathbb{Z}) \longrightarrow 0$$

where the maps are obtained by Poincaré Duality from the natural push-forwards on homology induced by the inclusions. We have (see, for instance, [1], 0.1.3, page 312)

$$H^{p+2}(X_l'', \mathbb{Z}) \cong H^{p+2}(X_l', \mathbb{Z}) \oplus H^p(Q_l'', \mathbb{Z}), \tag{2}$$

$$H^{p+2}(X_l', \mathbb{Z}) \cong H^{p+2}(X_l, \mathbb{Z}) \oplus \left(\oplus_{\substack{p-6 \leq i \leq p \\ i \equiv p[2]}} H^i(\pi^{-1}(T_l), \mathbb{Z}) \right), \tag{3}$$

$$H^{p+2}(X_l, \mathbb{Z}) \cong H^{p+2}(X, \mathbb{Z}) \oplus \left(\oplus_{\substack{p-2(n-4) \leq i \leq p \\ i \equiv p[2]}} H^i(l, \mathbb{Z}) \right) \tag{4}$$

and

$$H^{p-2}(S_l'', \mathbb{Z}) \cong H^{p-2}(S_l', \mathbb{Z}) \oplus H^{p-4}(R_l', \mathbb{Z}). \tag{5}$$

Since E_3 and L_l are $\mathbb{P}^1$-bundles over Q_l'' and S_l' respectively,

$$H^p(E_3, \mathbb{Z}) \cong H^p(Q_l'', \mathbb{Z}) \oplus H^{p-2}(Q_l'', \mathbb{Z}) \tag{6}$$

and

$$H^p(L_l, \mathbb{Z}) \cong H^p(S_l', \mathbb{Z}) \oplus H^{p-2}(S_l', \mathbb{Z}). \tag{7}$$

The map ψ' is the composition of the inclusion $H^{n-3}(S_l', \mathbb{Z}) \hookrightarrow H^{n-3}(L_l, \mathbb{Z})$ obtained from (7) with the differential $E_2^{n-3,1} \to E_2^{n-1,0}$ and the projection $H^{n-1}(X_l'', \mathbb{Z}) \twoheadrightarrow H^{n-1}(X, \mathbb{Z})$ obtained from (2), (3) and (4). We first study the cokernel of the differential $E_2^{n-3,1} \to E_2^{n-1,0}$.

By [8], 3.2.13, the differentials $E_3^{p,q} \to E_3^{p+3,q-2}$ are zero. Therefore $E_\infty^{\cdot,\cdot} = E_3^{\cdot,\cdot}$ and, in particular,

$$Coker \left(H^{n-3}(L_l, \mathbb{Z}) \oplus H^{n-3}(E_3, \mathbb{Z}) \longrightarrow H^{n-1}(X_l'', \mathbb{Z}) \right) =$$

$$= Coker \left(E_2^{n-3,1} \longrightarrow E_2^{n-1,0} \right)$$

$$= E_3^{n-1,0} = E_\infty^{n-1,0} = Gr^{n-1}(H^{n-1}(U_l, \mathbb{Z})).$$

This is the image of $H^{n-1}(X_l'', \mathbb{Z})$ in $H^{n-1}(U_l, \mathbb{Z})$ and, by [8] 3.2.17, it is the piece $W_{n-1}(H^{n-1}(U_l, \mathbb{Z}))$ of weight $n-1$ of the mixed Hodge structure on $H^{n-1}(U_l, \mathbb{Z})$.

Define $V_l := \mathbb{P}^{n-2'} \setminus Q_l'$. The fibers of the conic-bundle $U_l \to V_l$ are all smooth, hence

$$H^{n-1}(U_l, \mathbb{Z}) \cong H^{n-3}(V_l, \mathbb{Z}) \oplus H^{n-1}(V_l, \mathbb{Z})$$

Claim 5.2. *Under this isomorphism, the space $W_{n-1}(H^{n-1}(U_l, \mathbb{Z}))$ is isomorphic to $W_{n-3}(H^{n-3}(V_l, \mathbb{Z})) \oplus W_{n-1}(H^{n-1}(V_l, \mathbb{Z}))$.*

To prove this it is sufficient to show that the maps $H^{n-1}(V_l,\mathbb{Z})\to H^{n-1}(U_l,\mathbb{Z})$ and $H^{n-3}(V_l,\mathbb{Z})\to H^{n-1}(U_l,\mathbb{Z})$ are morphisms of mixed Hodge structures of type $(0,0)$ and $(1,1)$ respectively.

By [8] pages 37-38, the pull-backs on cohomology $H^{n-3}(V_l,\mathbb{Z})\to H^{n-3}(U_l,\mathbb{Z})$ and $H^{n-1}(V_l,\mathbb{Z})\to H^{n-1}(U_l,\mathbb{Z})$ are morphisms of mixed Hodge structures of type $(0,0)$. To see that the map $H^{n-3}(V_l,\mathbb{Z})\to H^{n-1}(U_l,\mathbb{Z})$ is a morphism of mixed Hodge structures of type $(1,1)$ choose a bisection B of the conic bundle $U_l\to V_l$ and let η be a half of the cohomology class of B. Then the map

$$H^{n-3}(V_l,\mathbb{Z})\longrightarrow H^{n-1}(U_l,\mathbb{Z})$$

is the composition of pull-back

$$H^{n-3}(V_l,\mathbb{Z})\longrightarrow H^{n-3}(U_l,\mathbb{Z})$$

with cup-product with η

$$H^{n-3}(U_l,\mathbb{Z})\longrightarrow H^{n-1}(U_l,\mathbb{Z})\,.$$

The class 2η is the restriction to U_l of the cohomology class of the closure of B in X'_l. Therefore 2η is in the image of

$$H^2(X'_l,\mathbb{Z})\longrightarrow H^2(U_l,\mathbb{Z})$$

and hence has pure weight 2 and Hodge type $(1,1)$. Therefore η has pure weight 2 and Hodge type $(1,1)$ in the mixed Hodge structure on $H^2(U_l,\mathbb{Z})$, the map $H^{n-3}(V_l,\mathbb{Z})\to H^{n-1}(U_l,\mathbb{Z})$ is a morphism of mixed Hodge structures of type $(1,1)$ and sends $W_{n-3}(H^{n-3}(V_l,\mathbb{Z}))$ into $W_{n-1}(H^{n-1}(U_l,\mathbb{Z}))$.

We now determine $W_{n-3}(H^{n-3}(V_l,\mathbb{Z}))\oplus W_{n-1}(H^{n-1}(V_l,\mathbb{Z}))$. In the following we let p be equal to $n-3$ or $n-1$.

Let $\mathbb{P}^{n-2''}\to\mathbb{P}^{n-2'}$ be the blow up of $\mathbb{P}^{n-2'}$ along R'_l with exceptional divisor E'' and identify Q''_l with its image in $\mathbb{P}^{n-2''}$. Then $V_l=\mathbb{P}^{n-2''}\setminus(E''\cup Q''_l)$ and the divisors E'' and Q''_l are smooth and meet transversally. Therefore $W_p(H^p(V_l,\mathbb{Z}))$ is the image of $H^p(\mathbb{P}^{n-2''},\mathbb{Z})$ in $H^p(V_l,\mathbb{Z})$, i.e., it is isomorphic to the cokernel of the map

$$H^{p-2}(Q''_l,\mathbb{Z})\oplus H^{p-2}(E'',\mathbb{Z})\longrightarrow H^p(\mathbb{P}^{n-2''},\mathbb{Z})$$

obtained by Poincaré Duality from push-forward on homology. Since E'' is a $\mathbb{P}^2$-bundle over R'_l, we have

$$H^{p-2}(E'',\mathbb{Z})\cong H^{p-2}(R'_l,\mathbb{Z})\oplus H^{p-4}(R'_l,\mathbb{Z})\oplus H^{p-6}(R'_l,\mathbb{Z})\,,\tag{8}$$

By e.g. [1] 0.1.3, we have the isomorphism

$$H^p(\mathbb{P}^{n-2''},\mathbb{Z}) \cong H^p(\mathbb{P}^{n-2'},\mathbb{Z}) \oplus H^{p-2}(R_l',\mathbb{Z}) \oplus H^{p-4}(R_l',\mathbb{Z})\,.$$

Under the map $H^{p-2}(E'',\mathbb{Z}) \to H^p(\mathbb{P}^{n-2''},\mathbb{Z})$ above, the summand $H^{p-2}(R_l',\mathbb{Z}) \oplus H^{p-4}(R_l',\mathbb{Z})$ in $H^{p-2}(E'',\mathbb{Z})$ maps isomorphically onto the same summand in $H^p(\mathbb{P}^{n-2''},\mathbb{Z})$. Therefore $W_p(H^p(V_l,\mathbb{Z}))$ is a quotient of $H^p(\mathbb{P}^{n-2'},\mathbb{Z})$.

The summand $H^{p-6}(R_l',\mathbb{Z})$ in $H^{p-2}(E'',\mathbb{Z})$ maps into the summand $H^p(\mathbb{P}^{n-2'},\mathbb{Z})$ of $H^p(\mathbb{P}^{n-2''},\mathbb{Z})$, the map $H^{p-6}(R_l',\mathbb{Z}) \to H^p(\mathbb{P}^{n-2'},\mathbb{Z})$ being again obtained by Poincaré Duality from push-forward on homology. Since the degree of R_l in $\mathbb{P}^{n-2}$ is 16, the image of the composition of $H^{p-6}(R_l',\mathbb{Z}) \hookrightarrow H^p(\mathbb{P}^{n-2'},\mathbb{Z})$ with the isomorphism

$$H^p(\mathbb{P}^{n-2'},\mathbb{Z}) \cong H^p(\mathbb{P}^{n-2},\mathbb{Z}) \oplus \left(\oplus_{\substack{p-10\le i\le p-2\\ i\equiv p[2]}} H^i(T_l,\mathbb{Z})\right)$$

contains an element whose component in the summand $H^p(\mathbb{P}^{n-2},\mathbb{Z})$ is 16 times a generator of $H^p(\mathbb{P}^{n-2},\mathbb{Z})$.

Since the degree of Q_l is 5, the image of the composition of the direct sum embedding

$$H^{p-2}(Q_l'',\mathbb{Z}) \hookrightarrow H^{p-2}(E'',\mathbb{Z}) \oplus H^{p-2}(Q_l'',\mathbb{Z})$$

with the map

$$H^{p-2}(E'',\mathbb{Z}) \oplus H^{p-2}(Q_l'',\mathbb{Z}) \longrightarrow H^p(\mathbb{P}^{n-2''},\mathbb{Z})$$

contains an element whose component in the summand $H^p(\mathbb{P}^{n-2},\mathbb{Z})$ is 5 times a generator of $H^p(\mathbb{P}^{n-2},\mathbb{Z})$. Since 16 and 5 are coprime, we deduce that the image of $H^{p-2}(E'',\mathbb{Z}) \oplus H^{p-2}(Q_l'',\mathbb{Z})$ in $H^p(\mathbb{P}^{n-2''},\mathbb{Z})$ contains an element whose component in the summand $H^p(\mathbb{P}^{n-2},\mathbb{Z})$ is a generator of $H^p(\mathbb{P}^{n-2},\mathbb{Z})$.

So far we have obtained that $W_p(H^p(V_l,\mathbb{Z}))$ *is a quotient of*

$$\oplus_{\substack{p-10\le i\le p-2\\ i\equiv p[2]}} H^i(T_l,\mathbb{Z}) \subset H^p(\mathbb{P}^{n-2''},\mathbb{Z})\,.$$

It is now easily seen that $\left(\oplus_{\substack{n-11\le i\le n-3\\ i\equiv n-1[2]}} H^i(T_l,\mathbb{Z})\right) \oplus \left(\oplus_{\substack{n-13\le i\le n-5\\ i\equiv n-1[2]}} H^i(T_l,\mathbb{Z})\right)$ maps into the summand

$$\oplus_{\substack{n-9\le i\le n-3\\ i\equiv n-3[2]}} H^i(\pi^{-1}(T_l),\mathbb{Z})$$

of $H^{n-1}(X_l'',\mathbb{Z})$. Therefore $W_{n-1}(H^{n-1}(U_l,\mathbb{Z})) = W_{n-3}(H^{n-3}(V_l,\mathbb{Z})) \oplus W_{n-1}(H^{n-1}(V_l,\mathbb{Z}))$ is a subquotient of

$$\oplus_{\substack{n-9\le i\le n-3\\ i\equiv n-3[2]}} H^i(\pi^{-1}(T_l),\mathbb{Z}) \subset H^{n-1}(X_l'',\mathbb{Z}) =$$

$$H^{n-1}(X_l,\mathbb{Z}) \oplus H^{n-3}(Q_l'',\mathbb{Z}) \oplus \left(\oplus_{\substack{n-9\le i\le n-3\\ i\equiv n-3[2]}} H^i(\pi^{-1}(T_l),\mathbb{Z})\right)$$

and the map

$$H^{n-3}(L_l,\mathbb{Z})\oplus H^{n-3}(E_3,\mathbb{Z})\longrightarrow H^{n-1}(X_l,\mathbb{Z})$$

is onto. So, in particular, we have proved

Claim 5.3. *The map*

$$H^{n-3}(L_l,\mathbb{Z})\oplus H^{n-3}(E_3,\mathbb{Z})\longrightarrow H^{n-1}(X,\mathbb{Z})$$

is onto.

Since E_3 is the exceptional divisor of the blow up $X_l''\to X_l'$, the image of

$$H^{n-3}(E_3,\mathbb{Z})\longrightarrow H^{n-1}(X,\mathbb{Z})$$

is equal to the image of

$$H^{n-5}(Q_l'',\mathbb{Z})\longrightarrow H^{n-1}(X,\mathbb{Z})\,.$$

We will prove that the image of this map is algebraic. Since $H^{n-1}(X,\mathbb{Z})$ is torsion-free, it is enough to prove this after tensoring with $\mathbb{Q}$. Since, by Poincaré Duality, $H^{n-5}(Q_l'',\mathbb{Q})\cong H^{n-1}(Q_l'',\mathbb{Q})^*$, we first determine $H^{n-1}(Q_l'',\mathbb{Q})$. For this we use the spectral sequence

$$E_2^{p,q}=H^p(\mathbb{P}^{n-2''},R^qu_*\mathbb{Z})\implies H^{p+q}(W,\mathbb{Z})$$

where $W:=\mathbb{P}^{n-2}\setminus Q_l=\mathbb{P}^{n-2''}\setminus\left(\widetilde{E}'\cup E''\cup Q_l''\right)$ with $\widetilde{E}'$ the proper transform of E' in $\mathbb{P}^{n-2''}$ and $u:W\hookrightarrow\mathbb{P}^{n-2''}$ is the inclusion. Recall that such a spectral sequence degenerates at E_3 ([8], 3.2.13). By [9] pages 23-24, we have $H^i(W,\mathbb{Z})=0$ for $i>dim(W)=n-2$. Therefore we obtain the following exact sequence from the spectral sequence

$$\begin{aligned}H^{n-5}(\widetilde{E}'\cap E''\cap Q_l'',\mathbb{Z})\xrightarrow{d_{n-3}} H^{n-3}(\widetilde{E}'\cap E'',\mathbb{Z})\oplus H^{n-3}(\widetilde{E}'\cap Q_l'',\mathbb{Z})\oplus H^{n-3}(E''\cap Q_l'',\mathbb{Z})\xrightarrow{d_{n-1}}\\ \xrightarrow{d_{n-1}} H^{n-1}(\widetilde{E}',\mathbb{Z})\oplus H^{n-1}(E'',\mathbb{Z})\oplus H^{n-1}(Q_l'',\mathbb{Z})\xrightarrow{d_{n+1}} H^{n+1}(\mathbb{P}^{n-2''},\mathbb{Z})\longrightarrow 0\ .\qquad (9)\end{aligned}$$

We have

Lemma 5.4. *The varieties whose cohomologies appear in sequence (9) are described as follows.*

$\widetilde{\mathbf{E}}'\cap\mathbf{E}''\cap\mathbf{Q}_l''$: *$\mathbb{P}^1$-bundle over $\mathcal{V}_l$ where $\mathcal{V}_l:=E'\cap R_l'$. The variety $\mathcal{V}_l$ is a $\mathbb{P}^2$-bundle over T_l and each of its fibers over T_l embeds into the corresponding fiber of E' as the Veronese surface. Hence*

$$H^{n-5}(\widetilde{E}'\cap E''\cap Q_l'',\mathbb{Z})\cong H^{n-5}(\mathcal{V}_l,\mathbb{Z})\oplus H^{n-7}(\mathcal{V}_l,\mathbb{Z})$$

and

$$H^i(\mathcal{V}_l,\mathbb{Z})\cong H^i(T_l,\mathbb{Z})\oplus H^{i-2}(T_l,\mathbb{Z})\oplus H^{i-4}(T_l,\mathbb{Z})\ .$$

$\mathbf{T}''_{\mathbf{l}} := \widetilde{\mathbf{E}}' \cap \mathbf{Q}''_{\mathbf{l}}$: *bundle over T_l with fibers isomorphic to the blow up $\widehat{S}^2\mathbb{P}^2$ of the symmetric square $S^2\mathbb{P}^2$ of $\mathbb{P}^2$ along the diagonal of $S^2\mathbb{P}^2$. A fiber of $\widetilde{E}' \cap E'' \cap Q''_l$ embeds into the corresponding fiber of $\widetilde{E}' \cap Q''_l$ as the exceptional divisor of the blow up $\widehat{S}^2\mathbb{P}^2 \to S^2\mathbb{P}^2$. We have*

$$H^{n-3}(T''_l,\mathbb{Z}) \cong H^{n-3}(T_l,\mathbb{Z}) \oplus H^{n-5}(T_l,\mathbb{Z}) \oplus H^{n-7}(T_l,\mathbb{Z})^{\oplus 2} \oplus H^{n-9}(T_l,\mathbb{Z}) \oplus H^{n-11}(T_l,\mathbb{Z}) \oplus H^{n-5}(\mathcal{V}_l,\mathbb{Z})$$

$$\cong H^{n-3}(T_l,\mathbb{Z}) \oplus H^{n-5}(T_l,\mathbb{Z}) \oplus H^{n-7}(T_l,\mathbb{Z}) \oplus H^{n-7}(\mathcal{V}_l,\mathbb{Z}) \oplus H^{n-5}(\mathcal{V}_l,\mathbb{Z})$$

and, under d_{n-3}, the summand $H^{n-7}(\mathcal{V}_l,\mathbb{Z}) \oplus H^{n-5}(\mathcal{V}_l,\mathbb{Z})$ in $H^{n-5}(\widetilde{E}' \cap E'' \cap Q''_l,\mathbb{Z})$ maps into the same summand in $H^{n-3}(T''_l,\mathbb{Z})$.

$\mathbf{E}'' \cap \mathbf{Q}''_{\mathbf{l}}$: *$\mathbb{P}^1$-bundle over R'_l. Hence*

$$H^{n-3}(E'' \cap Q''_l,\mathbb{Z}) \cong H^{n-3}(R'_l,\mathbb{Z}) \oplus H^{n-5}(R'_l,\mathbb{Z}) \ .$$

$\widetilde{\mathbf{E}}' \cap \mathbf{E}''$: *$\mathbb{P}^2$-bundle over $\mathcal{V}_l$ which contains $\widetilde{E}' \cap E'' \cap Q''_l$ as a conic-bundle over $\mathcal{V}_l$. We have*

$$H^{n-3}(\widetilde{E}' \cap E'',\mathbb{Z}) \cong H^{n-3}(\mathcal{V}_l,\mathbb{Z}) \oplus H^{n-5}(\mathcal{V}_l,\mathbb{Z}) \oplus H^{n-7}(\mathcal{V}_l,\mathbb{Z}) \ .$$

$\widetilde{\mathbf{E}}'$: *the blow up of E' along $\mathcal{V}_l$, i.e., bundle over T_l with fibers isomorphic to the blow up of $\mathbb{P}^5$ along the Veronese surface. This contains $\widetilde{E}' \cap E''$ as its exceptional divisor. Hence*

$$H^{n-1}(\widetilde{E}',\mathbb{Z}) \cong H^{n-3}(\mathcal{V}_l,\mathbb{Z}) \oplus H^{n-5}(\mathcal{V}_l,\mathbb{Z}) \oplus H^{n-1}(T_l,\mathbb{Z}) \oplus$$

$$\oplus H^{n-3}(T_l,\mathbb{Z}) \oplus H^{n-5}(T_l,\mathbb{Z}) \oplus H^{n-7}(T_l,\mathbb{Z}) \oplus H^{n-9}(T_l,\mathbb{Z}) \oplus H^{n-11}(T_l,\mathbb{Z}) \ .$$

$\mathbf{E}''$: *$\mathbb{P}^2$-bundle over R'_l which contains $E'' \cap Q''_l$ as a conic-bundle over R'_l. Hence*

$$H^{n-1}(E'',\mathbb{Z}) \cong H^{n-1}(R'_l,\mathbb{Z}) \oplus H^{n-3}(R'_l,\mathbb{Z}) \oplus H^{n-5}(R'_l,\mathbb{Z}) \ .$$

Proof : Easy. ☐

Lemma 5.5. *There is a natural exact sequence*

$$0 \longrightarrow H^{n-3}(T_l,\mathbb{Q}) \oplus H^{n-5}(T_l,\mathbb{Q}) \oplus H^{n-7}(T_l,\mathbb{Q})^{\oplus 2} \oplus H^{n-9}(T_l,\mathbb{Q}) \oplus H^{n-3}(R'_l,\mathbb{Q}) \longrightarrow$$

$$\longrightarrow H^{n-1}(Q''_l,\mathbb{Q}) \longrightarrow H^{n+1}(\mathbb{P}^{n-2},\mathbb{Q}) \longrightarrow 0$$

where the map

$$H^{n-3}(T_l,\mathbb{Q}) \oplus H^{n-5}(T_l,\mathbb{Q}) \oplus H^{n-7}(T_l,\mathbb{Q})^{\oplus 2} \oplus H^{n-9}(T_l,\mathbb{Q}) \longrightarrow H^{n-1}(Q''_l,\mathbb{Q})$$

is obtained from the inclusion $T''_l \subset Q''_l$.

Proof : From the description of $\widetilde{E}' \cap Q_l''$ in Lemma 5.4 it follows that the map d_{n-3} in sequence (9) is injective and we have the exact sequence

$$0 \longrightarrow H^{n-5}(\widetilde{E}' \cap E'' \cap Q_l'', \mathbb{Z}) \xrightarrow{d_{n-3}} H^{n-3}(\widetilde{E}' \cap E'', \mathbb{Z}) \oplus H^{n-3}(\widetilde{E}' \cap Q_l'', \mathbb{Z}) \oplus H^{n-3}(E'' \cap Q_l'', \mathbb{Z}) \xrightarrow{d_{n-1}}$$

$$\xrightarrow{d_{n-1}} H^{n-1}(\widetilde{E}', \mathbb{Z}) \oplus H^{n-1}(E'', \mathbb{Z}) \oplus H^{n-1}(Q_l'', \mathbb{Z}) \xrightarrow{d_{n+1}} H^{n+1}(\mathbb{P}^{n-2''}, \mathbb{Z}) \longrightarrow 0 \ .$$

Tensoring the exact sequence (9) with $\mathbb{Q}$ and using Lemma 5.4 and the isomorphism

$$H^{n+1}(\mathbb{P}^{n-2''}, \mathbb{Z}) \cong H^{n+1}(\mathbb{P}^{n-2}, \mathbb{Z}) \oplus$$

$$\oplus H^{n-1}(T_l, \mathbb{Z}) \oplus H^{n-3}(T_l, \mathbb{Z}) \oplus H^{n-5}(T_l, \mathbb{Z}) \oplus H^{n-7}(T_l, \mathbb{Z}) \oplus H^{n-9}(T_l, \mathbb{Z}) \oplus$$

$$\oplus H^{n-1}(R_l', \mathbb{Z}) \oplus H^{n-3}(R_l', \mathbb{Z}) \ ,$$

we easily deduce Lemma 5.5. □

Remark 5.6. *In fact we have the exact sequence*

$$0 \longrightarrow H^{n-3}(T_l, \mathbb{Z}[\frac{1}{30}]) \oplus H^{n-5}(T_l, \mathbb{Z}[\frac{1}{30}]) \oplus H^{n-7}(T_l, \mathbb{Z}[\frac{1}{30}])^{\oplus 2} \oplus H^{n-9}(T_l, \mathbb{Z}[\frac{1}{30}]) \oplus H^{n-3}(R_l', \mathbb{Z}[\frac{1}{30}])$$

$$\longrightarrow H^{n-1}(Q_l'', \mathbb{Z}[\frac{1}{30}]) \longrightarrow H^{n+1}(\mathbb{P}^{n-2}, \mathbb{Z}[\frac{1}{30}]) \longrightarrow 0 \ .$$

It follows from the previous lemma (since the cohomology of X has no torsion) that the image of

$$H^{n-5}(Q_l'', \mathbb{Z}) \longrightarrow H^{n-1}(X, \mathbb{Z})$$

is algebraic. Hence the image of the composition $H^{n-5}(Q_l'', \mathbb{Z}) \to H^{n-1}(X, \mathbb{Z}) \twoheadrightarrow H^{n-1}(X, \mathbb{Z})^0$ is algebraic. For X generic $H^{n-1}(X, \mathbb{Z})^0$ has no nonzero algebraic part. Hence for X generic and therefore, for all X, the image of $H^{n-5}(Q_l'', \mathbb{Z}) \to H^{n-1}(X, \mathbb{Z})^0$ is zero. Hence the map

$$H^{n-3}(L_l, \mathbb{Z}) \longrightarrow H^{n-1}(X, \mathbb{Z})^0$$

is onto. We have

$$H^{n-3}(L_l, \mathbb{Z}) \cong H^{n-3}(S_l', \mathbb{Z}) \oplus H^{n-5}(S_l', \mathbb{Z})$$

and the restriction $H^{n-5}(S_l', \mathbb{Z}) \to H^{n-1}(X, \mathbb{Z})^0$ is the composition of pull-back $H^{n-5}(S_l', \mathbb{Z}) \to H^{n-5}(S_l'', \mathbb{Z})$, and push-forward $H^{n-5}(S_l'', \mathbb{Z}) \to H^{n-5}(Q_l'', \mathbb{Z}) \to H^{n-1}(X, \mathbb{Z})^0$. Hence the map $H^{n-5}(S_l', \mathbb{Z}) \to H^{n-1}(X, \mathbb{Z})^0$ is zero and the map

$$H^{n-3}(S_l', \mathbb{Z}) \longrightarrow H^{n-1}(X, \mathbb{Z})^0$$

is onto.

Now, we have

$$H^{n-3}(S'_l,\mathbb{Z}) \cong H^{n-3}(S_l,\mathbb{Z}) \oplus H^{n-5}(\mathcal{P}^*_l,\mathbb{Z}) \oplus H^{n-7}(\mathcal{P}^*_l,\mathbb{Z})\,.$$

recall that $\mathcal{P}^*_l$ is the variety parametrizing lines in the fibers of $\pi^{-1}(T_l) \to T_l$. Therefore $\mathcal{P}^*_l$ is a $\mathbb{P}^2$-bundle over T_l. Using the fact that T_l is a smooth complete intersection of dimension $n-8$ in $\mathbb{P}^{n-2}$, one immediately sees that the image of the summand $H^{n-5}(\mathcal{P}^*_l,\mathbb{Z}) \oplus H^{n-7}(\mathcal{P}^*_l,\mathbb{Z})$ of $H^{n-3}(S'_l,\mathbb{Z})$ in $H^{n-1}(X,\mathbb{Z})^0$ is zero. Therefore the map

$$H^{n-3}(S_l,\mathbb{Z}) \longrightarrow H^{n-1}(X,\mathbb{Z})^0$$

is onto. This proves the theorem in the case where n is even, since in that case $H^{n-1}(X,\mathbb{Z})^0 = H^{n-1}(X,\mathbb{Z})$.

Let σ_1 be the inverse image in S_l of the hyperplane class on the Grassmannian $G(2,n+1)$ by the composition $S_l \to D_l \hookrightarrow G(2,n+1)$. If n is odd, one easily computes that the image of $\sigma_1^{(n-3)/2}$ in $H^{n-1}(X,\mathbb{Z})$ is $5\zeta^{(n-1)/2}$ where ζ is the hyperplane class on X. On the other hand, let x be a general point on l and let L_x be the union of the lines in X through x. Then L_x is the intersection of X with the hyperplane tangent to X at x and a quadric (it is the second osculating cone to X at x). The cohomology class of a linear section (through x) of L_x of codimension $\frac{n-1}{2}-2$ is $2\zeta^{(n-1)/2}$ in X and it is in the image of $H^{n-3}(S_l,\mathbb{Z})$. Since 2 and 5 are coprime, the image of $H^{n-3}(S_l,\mathbb{Z})$ in $H^{n-1}(X,\mathbb{Z})$ contains $\zeta^{(n-1)/2}$ and the map

$$\psi : H^{n-3}(S_l,\mathbb{Z}) \longrightarrow H^{n-1}(X,\mathbb{Z})$$

in onto for n odd as well. It is now immediate that ψ' is also onto for n odd. ☐

Let h be the first Chern class of the pull-back of $\mathcal{O}_{\mathbb{P}^{n-2}}(1)$ to S'_l, let σ_i be the pull-back to S'_l of the i-th Chern class of the universal quotient bundle on the grassmannian $G(2,n+1) \supset D_l$ and let e_2 be the first Chern class of the exceptional divisor of $S'_l \to S_l$. We make the

Definition 5.7. *For a positive integer k the $k-th$ primitive cohomologies of S_l and S'_l are*

$$H^k(S_l,\mathbb{Z})^0 := (\mathbb{Z}h \oplus \mathbb{Z}\sigma_1)^{\perp} \subset H^k(S_l,\mathbb{Z})$$

and

$$H^k(S'_l,\mathbb{Z})^0 := (\mathbb{Z}h \oplus \mathbb{Z}\sigma_1 \oplus \mathbb{Z}e_2)^{\perp} \subset H^k(S'_l,\mathbb{Z})$$

where $\perp$ means orthogonal complement with respect to cup-product.

Composing the map ψ' with restriction to $H^{n-3}(S'_l,\mathbb{Z})^0$ on the right and with the projection $H^{n-1}(X,\mathbb{Z}) \twoheadrightarrow H^{n-1}(X,\mathbb{Z})^0$ on the left, we get $\psi'^0 : H^{n-3}(S'_l,\mathbb{Z})^0 \to H^{n-1}(X,\mathbb{Z})^0$. Our goal is to prove the following generalization of the results of Clemens and Griffiths.

Theorem 5.8. *The map ψ'^0 is onto and its kernel is the i_l-invariant part $H^{n-3}(S'_l,\mathbb{Z})^{0+}$ of $H^{n-3}(S'_l,\mathbb{Z})^0$.*

The first step for proving the theorem is

Theorem 5.9. *Let a and b be two elements of $H^{n-3}(S'_l,\mathbb{Z})^0$. Then*

$$\psi'(a).\psi'(b) = a.i_l^*b - a.b\,.$$

Proof : We have

$$\psi'(a).\psi'(b) = (\epsilon_1\epsilon_2\rho)_*p^*a.(\epsilon_1\epsilon_2\rho)_*p^*b = (\epsilon_2\rho)_*p^*a.\epsilon_1^*\epsilon_{1*}(\epsilon_2\rho)_*p^*b\,.$$

Let ξ_1 be the first Chern class of the tautological invertible sheaf for the projective bundle $g_1 : E_1 \to l$. Let γ_i^1 be the Chern classes of the universal quotient bundle on the projective bundle $g_1 : E_1 \to l$, i.e.,

$$\gamma_i^1 = \xi_1^i + \xi_1^{i-1}.g_1^*c_1(N_{l/X}) + \ldots + g_1^*c_i(N_{l/X})\,.$$

Define ξ_2, γ_i^2 and ξ_3, γ_i^3 similarly for the projective bundles $g_2 : E_2 \to \pi^{-1}(T_l)$ and $g_3 : E_3 \to Q''_l$ respectively. By, e.g., [1], 0.1.3, we have

$$\epsilon_1^*\epsilon_{1*}(\epsilon_2\rho)_*p^*b = (\epsilon_2\rho)_*p^*b + i_{1*}\left(\sum_{r=0}^{n-4}\xi_1^r.g_1^*g_{1*}\left(\gamma_{n-4-r}^1.i_1^*\left((\epsilon_2\rho)_*p^*b\right)\right)\right)$$

where $i_1 : E_1 \hookrightarrow X_l$ is the inclusion. We also let $i_2 : E_2 \hookrightarrow X'_l$ and $i_3 : E_3 \hookrightarrow X''_l$ be the inclusions.

For any r, $(0 \le r \le n-4)$, we have

$$g_{1*}\left(\gamma_{n-4-r}.i_1^*(\epsilon_2\rho)_*p^*b\right) \in H^{n-3-2r}(l,\mathbb{Z})\,.$$

Therefore $g_{1*}\left(\gamma_{n-4-r}.i_1^*(\epsilon_2\rho)_*p^*b\right) \neq 0$ only if $n-3-2r=0$ or $n-3-2r=2$. This is impossible if n is even so *we now suppose that n is odd.* So if we put

$$B := i_{1*}\left(\xi_1^{(n-3)/2}.g_1^*g_{1*}\left(\gamma_{(n-5)/2}^1.i_1^*\left((\epsilon_2\rho)_*p^*b\right)\right) + \xi_1^{(n-5)/2}.g_1^*g_{1*}\left(\gamma_{(n-3)/2}^1.i_1^*\left((\epsilon_2\rho)_*p^*b\right)\right)\right)\,,$$

we have

$$\epsilon_1^*\epsilon_{1*}(\epsilon_2\rho)_*p^*b = (\epsilon_2\rho)_*p^*b + B\,.$$

If $n \geq 7$, replacing $\gamma^1_{(n-5)/2}$ and $\gamma^1_{(n-3)/2}$ in terms of ξ_1, we obtain

$$B = i_{1*}\left(\xi_1^{(n-3)/2}.g_1^*g_{1*}\left(\xi_1^{(n-5)/2}.i_1^*\left((\epsilon_2\rho)_*p^*b\right) + \xi_1^{(n-7)/2}.g_1^*c_1(N_{l/X}).i_1^*\left((\epsilon_2\rho)_*p^*b\right)\right)\right) +$$

$$+i_{1*}\left(\xi_1^{(n-5)/2}.g_1^*g_{1*}\left(\xi_1^{(n-3)/2}.i_1^*\left((\epsilon_2\rho)_*p^*b\right) + \xi_1^{(n-5)/2}.g_1^*c_1(N_{l/X}).i_1^*\left((\epsilon_2\rho)_*p^*b\right)\right)\right) .$$

We have $c_1(N_{l/X}) = (n-4)j_1^*\zeta$ where $\zeta = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ and $j_1 : l \hookrightarrow X$ is the inclusion. Similarly we define $j_2 : \pi^{-1}(T_l) \hookrightarrow X_l$ and $j_3 : Q_l'' \hookrightarrow X_l'$ to be the inclusions. Therefore we obtain

$$B = i_{1*}\left(\xi_1^{(n-3)/2}.g_1^*g_{1*}\left(\xi_1^{(n-5)/2}.i_1^*\left((\epsilon_2\rho)_*p^*b\right) + \xi_1^{(n-7)/2}.(n-4)g_1^*j_1^*\zeta.i_1^*\left((\epsilon_2\rho)_*p^*b\right)\right)\right) +$$

$$+i_{1*}\left(\xi_1^{(n-5)/2}.g_1^*g_{1*}\left(\xi_1^{(n-3)/2}.i_1^*\left((\epsilon_2\rho)_*p^*b\right) + \xi_1^{(n-5)/2}.(n-4)g_1^*j_1^*\zeta.i_1^*\left((\epsilon_2\rho)_*p^*b\right)\right)\right) .$$

Or, since $j_1g_1 = \epsilon_1 i_1$,

$$B = i_{1*}\left(\xi_1^{(n-3)/2}.g_1^*g_{1*}\left(\xi_1^{(n-5)/2}.i_1^*\left((\epsilon_2\rho)_*p^*b\right) + \xi_1^{(n-7)/2}.(n-4)i_1^*\epsilon_1^*\zeta.i_1^*\left((\epsilon_2\rho)_*p^*b\right)\right)\right) +$$

$$+i_{1*}\left(\xi_1^{(n-5)/2}.g_1^*g_{1*}\left(\xi_1^{(n-3)/2}.i_1^*\left((\epsilon_2\rho)_*p^*b\right) + \xi_1^{(n-5)/2}.(n-4)i_1^*\epsilon_1^*\zeta.i_1^*\left((\epsilon_2\rho)_*p^*b\right)\right)\right) .$$

Let E_1 also denote the first Chern class of the invertible sheaf $\mathcal{O}_{X_l}(E_1)$. Since $\xi_1 = -i_1^*E_1$, we can write

$$B = (-1)^n i_{1*}\left(i_1^*E_1^{(n-3)/2}.g_1^*g_{1*}i_1^*\left(E_1^{(n-5)/2}.\left((\epsilon_2\rho)_*p^*b\right) - E_1^{(n-7)/2}.(n-4)\epsilon_1^*\zeta.\left((\epsilon_2\rho)_*p^*b\right)\right)\right) +$$

$$+(-1)^n i_{1*}\left(i_1^*E_1^{(n-5)/2}.g_1^*g_{1*}i_1^*\left(E_1^{(n-3)/2}.\left((\epsilon_2\rho)_*p^*b\right) - E_1^{(n-5)/2}.(n-4)\epsilon_1^*\zeta.\left((\epsilon_2\rho)_*p^*b\right)\right)\right) .$$

Or, since $g_{1*}i_1^* = j_1{}^*\epsilon_{1*}$,

$$B = (-1)^n i_{1*}\left(i_1^*E_1^{(n-3)/2}.g_1^*j_1{}^*\epsilon_{1*}\left(E_1^{(n-5)/2}.\left((\epsilon_2\rho)_*p^*b\right) - E_1^{(n-7)/2}.(n-4)\epsilon_1^*\zeta.\left((\epsilon_2\rho)_*p^*b\right)\right)\right) +$$

$$+(-1)^n i_{1*}\left(i_1^*E_1^{(n-5)/2}.g_1^*j_1{}^*\epsilon_{1*}\left(E_1^{(n-3)/2}.\left((\epsilon_2\rho)_*p^*b\right) - E_1^{(n-5)/2}.(n-4)\epsilon_1^*\zeta.\left((\epsilon_2\rho)_*p^*b\right)\right)\right) .$$

Now

$$\epsilon_{1*}\left(E_1^{(n-5)/2}.\left((\epsilon_2\rho)_*p^*b\right) - E_1^{(n-7)/2}.(n-4)\epsilon_1^*\zeta.\left((\epsilon_2\rho)_*p^*b\right)\right)$$

is an element of $H^{2n-6}(X, \mathbb{Z})$. Hence its image by j_1^* is zero unless $2n-6 \leq 2$, i.e., $n \leq 4$. We supposed $n \geq 7$. Similarly,

$$j_1^*\epsilon_{1*}\left(E_1^{(n-5)/2}.\left((\epsilon_2\rho)_*p^*b\right) - E_1^{(n-7)/2}.(n-4)\epsilon_1^*\zeta.\left((\epsilon_2\rho)_*p^*b\right)\right)$$

is zero unless $2n-4 \leq 2$ which implies $n \leq 3$. Hence B is zero for $n \geq 7$. Similarly, B is zero for $n = 5$.

Therefore

$$\psi'(a).\psi'(b) = (\epsilon_2\rho)_*p^*a.(\epsilon_2\rho)_*p^*b \,.$$

Now write

$$\psi'(a).\psi'(b) = \rho_* p^* a.\epsilon_2^* \epsilon_{2*} \rho_* p^* b$$

and, as before,

$$\epsilon_2^* \epsilon_{2*} \rho_* p^* b = \rho_* p^* b + i_{2*} \left(\sum_{r=0}^{3} \xi_2^r . g_2^* g_{2*} \left(\gamma_{3-r}^2 . i_2^* \rho_* p^* b \right) \right) .$$

So

$$\psi'(a).\psi'(b) = \rho_* p^* a.\rho_* p^* b + \rho_* p^* a.i_{2*} \left(\sum_{r=0}^{3} \xi_2^r . g_2^* g_{2*} \left(\gamma_{3-r}^2 . i_2^* \rho_* p^* b \right) \right)$$

or

$$\psi'(a).\psi'(b) = \rho_* p^* a.\rho_* p^* b + i_2^* \rho_* p^* a. \left(\sum_{r=0}^{3} \xi_2^r . g_2^* g_{2*} \left(\gamma_{3-r}^2 . i_2^* \rho_* p^* b \right) \right) .$$

We have $a.e_2 = 0$. Hence $p^* a.p^* e_2 = 0$. Let E_2 also denote the cohomology class of E_2. Then it is easily seen that $\rho^* E_2 = p^* e_2$. Therefore $p^* a.\rho^* E_2 = 0$. In order to use this, we need to modify the above expression a bit.

We first need to write the first three Chern classes of $N_{\pi^{-1}(T_l)/X_l}$ as inverse images of cohomology classes by j_2. Consider the exact sequence

$$0 \longrightarrow N_{\pi^{-1}(T_l)/X_l} \longrightarrow N_{\pi^{-1}(T_l)/\mathbb{P}_l^n} \longrightarrow N_{X_l/\mathbb{P}_l^n}|_{\pi^{-1}(T_l)} \longrightarrow 0 .$$

We have

$$N_{X_l/\mathbb{P}_l^n} \cong \mathcal{O}_{\mathbb{P}E}(2) \otimes \pi^* \mathcal{O}_{\mathbb{P}^{n-2}}(1)$$

where $E = \mathcal{O}_{\mathbb{P}^{n-2}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}^{\oplus 2}$, so that $\mathbb{P}E \cong \mathbb{P}_l^n$. Also

$$N_{\pi^{-1}(T_l)/\mathbb{P}_l^n} \cong \pi^* N_{T_l/\mathbb{P}^{n-2}} \cong \pi^* \left(\mathcal{O}_{\mathbb{P}^{n-2}}(3) \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(2)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^{n-2}}(1)^{\oplus 3} \right) .$$

It follows that we can write $c_i(N_{\pi^{-1}(T_l)/X_l}) = j_2^* c_i$ where the c_i are cohomology classes on X_l. So

$$\gamma_r^2 = \xi_2^r + \xi_2^{r-1}.g_2^* j_2^* c_1 + \ldots + g_2^* j_2^* c_r$$

and, since $\xi_2 = -i_2^* E_2$ and $j_2 g_2 = \epsilon_2 i_2$, we have

$$\gamma_r^2 = i_2^* \alpha_r^2$$

where

$$\alpha_r^2 = (-1)^r E_2^r + (-1)^{r-1} E_2^{r-1}.\epsilon_2^* c_1 + \ldots + \epsilon_2^* c_r .$$

Therefore, using $g_{2*} i_2^* = j_2^* \epsilon_{2*}$ and $j_2 g_2 = \epsilon_2 i_2$,

$$i_2^* \rho_* p^* a. \left(\sum_{r=0}^{3} \xi_2^r . g_2^* g_{2*} \left(\gamma_{3-r}^2 . i_2^* \rho_* p^* b \right) \right) = i_2^* \left(\rho_* p^* a. \left(\sum_{r=0}^{3} (-1)^r E_2^r . \epsilon_2^* \epsilon_{2*} \left(\alpha_{3-r}^2 . \rho_* p^* b \right) \right) \right)$$

$$= \rho_* p^* a.E_2. \left(\sum_{r=0}^{3} (-1)^r E_2^r . \epsilon_2^* \epsilon_{2*} \left(\alpha_{3-r}^2 . \rho_* p^* b \right) \right) = p^* a.\rho^* E_2.\rho^* \left(\sum_{r=0}^{3} (-1)^r E_2^r . \epsilon_2^* \epsilon_{2*} \left(\alpha_{3-r}^2 . \rho_* p^* b \right) \right) = 0$$

and we obtain

$$\psi'(a).\psi'(b) = \rho_* p^* a . \rho_* p^* b \,.$$

Writing $\rho = \epsilon_3 \widetilde{q}$, we have

$$\psi'(a).\psi'(b) = (\epsilon_3 \widetilde{q})_* p^* a . (\epsilon_3 \widetilde{q})_* p^* b = \widetilde{q}_* p^* a . \epsilon_3^* \epsilon_{3*} \widetilde{q}_* p^* b$$

and, as before,

$$\psi'(a).\psi'(b) = \widetilde{q}_* p^* a . \widetilde{q}_* p^* b + \widetilde{q}_* p^* a . i_{3*} g_3^* g_{3*} i_3^* \widetilde{q}_* p^* b = \widetilde{q}_* p^* a . \widetilde{q}_* p^* b + i_3^* \widetilde{q}_* p^* a . g_3^* g_{3*} i_3^* \widetilde{q}_* p^* b \,.$$

Consider the commutative diagram

$$\begin{tikzcd}
& S_l'' \arrow[r, "q'"] \arrow[dl, "\epsilon_4"'] \arrow[d, "i_3'"] & E_3 \arrow[r, "g_3"] \arrow[d, "i_3"] & Q_l'' \arrow[d, "j_3"] \\
S_l' & L_l \arrow[l, "p"'] \arrow[r, "\widetilde{q}"] & X_l'' \arrow[r, "\epsilon_3"] & X_l'
\end{tikzcd}$$

where the two squares are fiber squares. Using the diagram, we modify $\psi'(a).\psi'(b)$ as follows

$$\psi'(a).\psi'(b) = \widetilde{q}_* p^* a . \widetilde{q}_* p^* b + q'_* {i'_3}^* p^* a . g_3^* g_{3*} q'_* {i'_3}^* p^* b =$$

$$= \widetilde{q}_* p^* a . \widetilde{q}_* p^* b + q'_* {\epsilon_4}^* a . g_3^* g_{3*} q'_* {\epsilon_4}^* b = \widetilde{q}_* p^* a . \widetilde{q}_* p^* b + {\epsilon_4}^* a . (g_3 q')^* (g_3 q')_* {\epsilon_4}^* b \,.$$

The morphism $g_3 q' : S_l'' \to Q_l''$ is a double cover whose involution i_l' is the lift of i_l. Therefore

$$(g_3 q')^* (g_3 q')_* {\epsilon_4}^* b = \epsilon_4^* b + {i_l'}^* \epsilon_4^* b = \epsilon_4^* b + \epsilon_4^* i_l^* b$$

and

$$ {\epsilon_4}^* a . (g_3 q')^* (g_3 q')_* {\epsilon_4}^* b = \epsilon_4^* a . \left(\epsilon_4^* b + \epsilon_4^* i_l^* b \right) = a . \epsilon_{4*} \left(\epsilon_4^* b + \epsilon_4^* i_l^* b \right) = a . \left(b + i_l^* b \right) \;.$$

On the other hand

$$\widetilde{q}_* p^* a . \widetilde{q}_* p^* b = p^* a . p^* b . \widetilde{q}^* L_l$$

where we also denote by L_l the cohomology class of L_l in X_l''. We have the following

Lemma 5.10. *The cohomology class of L_l in X_l'' is equal to*

$$5 \left(\epsilon_1 \epsilon_2 \epsilon_3 \right)^* \zeta - 5 \left(\epsilon_2 \epsilon_3 \right)^* E_1 - 2 E_3 - k \epsilon_3^* E_2$$

for some nonnegative integer k.

Proof : To compute the coefficient of $(\epsilon_1\epsilon_2\epsilon_3)^*\zeta$, we push L_l forward to X and compute its degree in $\mathbb{P}^n$. The image of L_l in X is the union of all the lines in X which are incident to l. Since any such line maps to a point of Q_l by the projection from l, the image of L_l is the intersection with X of the cone of vertex l over Q_l. Since Q_l has degree 5, this proves that the coefficient of $(\epsilon_1\epsilon_2\epsilon_3)^*\zeta$ is 5.

The coefficient of $(\epsilon_2\epsilon_3)^* E_1$ is the negative of the multiplicity of the image of L_l in X along l. Intersecting X with a general linear subspace of dimension 3 which contains l, we see that this linear subspace contains 10 distinct lines which are distinct from l and are in the image of L_l. Therefore, the multiplicity of the image of L_l along l is exactly $5 = 5.3 - 10$.

The coefficient of E_3 is the negative of the multiplicity of the image of L_l in X'_l along Q''_l. This is 2 since L_l is smooth and ρ is an embedding outside S''_l and has degree 2 on S''_l. ☐

Now we will use the hypothesis $a.h = 0$. It implies $p^*a.p^*h = 0$. One easily sees that

$$p^*h = (\epsilon_2\rho)^*\,\pi_X^* c_1(\mathcal{O}_{\mathbb{P}^{n-2}}(1)) \, .$$

On the other hand $\epsilon_1^*\zeta - E_1 = \pi^* c_1(\mathcal{O}_{\mathbb{P}^{n-2}}(1))$. Therefore

$$p^*a.\,(\epsilon_1\epsilon_2\rho)^*\,\zeta = p^*a.\,(\epsilon_2\rho)^*\,E_1 \, .$$

Furthermore, we saw that $p^*a.\rho^*E_2 = 0$, hence,

$$\widetilde{q}_*p^*a.\widetilde{q}_*p^*b = p^*a.p^*b.\widetilde{q}^*L_l = p^*a.p^*b.\,(-2\widetilde{q}^*E_3) = -2a.b \ \ .$$

Finally,

$$\psi'(a).\psi'(b) = -2a.b + a.\,(b + i_l^*b) = a.i_l^*b - a.b \ \ .$$

☐

Corollary 5.11. *If ψ'^0 is onto, the kernel of ψ'^0 is equal to the set of i_l-invariant elements of $H^{n-3}(S'_l,\mathbb{Z})$.*

Proof : Let b be an element of $H^{n-3}(S'_l,\mathbb{Z})^0$. Then $\psi'^0(b)$ is zero if and only if,

$$\text{for every element } c \text{ of } H^{n-1}(X,\mathbb{Z})^0,\ \ \psi'(b).c = 0 \ \ .$$

If ψ'^0 is onto, this is equivalent to

$$\text{for every element } a \text{ of } H^{n-3}(S'_l,\mathbb{Z})^0,\ \ \psi'(a).\psi'(b) = 0 \ \ .$$

By theorem 5.9, this is equivalent to

$$\text{for every element } a \text{ of } H^{n-3}(S'_l,\mathbb{Z})^0,\ \ a.(i_l^*b-b)=0$$

which is in turn equivalent to

$$b=i_l^*b\ .$$

□

We are now ready to prove

Lemma 5.12. *Suppose* $n \geq 6$*, then*

$$H^2(S_l,\mathbb{Q})=\mathbb{Q}h\oplus\mathbb{Q}\sigma_1$$

$$H^2(S'_l,\mathbb{Q})=\mathbb{Q}h\oplus\mathbb{Q}\sigma_1\oplus\mathbb{Q}e_2$$

and, if $n=5$*, we have the exact sequence*

$$0\longrightarrow H^2(Q_l,\mathbb{Z})^0\longrightarrow H^2(S_l,\mathbb{Z})^0\longrightarrow H^4(X,\mathbb{Z})^0\longrightarrow 0$$

and

$$H^2(S_l,\mathbb{Q})=H^2(S_l,\mathbb{Q})^0\oplus\mathbb{Q}h\oplus\mathbb{Q}\sigma_1$$

(note that $T_l=\emptyset$ *for* $n\leq 7$ *so that* $Q_l=Q'_l$ *and* $S_l=S'_l$*).*

Proof : First suppose $n=5$. Then the direct sum decomposition above is clear. To prove the exactness of the sequence, note that $H^2(S_l,\mathbb{Z})\to H^4(X,\mathbb{Z})^0$ is onto by Theorem 5.1. Since $\mathbb{Z}h\oplus\mathbb{Z}\sigma_1$ is algebraic, its image in $H^4(X,\mathbb{Z})^0$ is algebraic. For X generic, the group $H^4(X,\mathbb{Z})^0$ has no nonzero algebraic part. Therefore for X generic and hence for all X, the image of $\mathbb{Z}h\oplus\mathbb{Z}\sigma_1$ in $H^4(X,\mathbb{Z})^0$ is zero. It follows that the sequence is exact on the right. The exactness of the rest of the sequence now follows from Corollary 5.11.

Now suppose $n\geq 6$. Since $H^2(S'_l,\mathbb{Q})\cong H^2(S_l,\mathbb{Q})\oplus\mathbb{Q}e_2$, we only need to compute $H^2(S_l,\mathbb{Q})$. Let H_1 be a general hyperplane in $\mathbb{P}^{n-2}$ and let H_2 be its inverse image in $\mathbb{P}^n$. The inverse image $S_{l,H}$ of H_1 in S_l parametrizes the lines in the fibers of $X_{l,H}\to H_1$ where $X_{l,H}$ is the proper transform of $X_H:=X\cap H_2$ in X_l. By [9] pages 23-25, we have $H^2(S_l,\mathbb{Z})\cong H^2(S_{l,H},\mathbb{Z})$ for $n\geq 7$ and $H^2(S_l,\mathbb{Z})\hookrightarrow H^2(S_{l,H},\mathbb{Z})$ for $n=6$. Suppose therefore that $n=6$. If we choose a general pencil of hyperplanes in $\mathbb{P}^{n-2}$ of which H_1 is a member, then $H^2(S_l,\mathbb{Z})$ maps into the part of $H^2(S_{l,H},\mathbb{Z})$ which is invariant under monodromy. Since $H^4(X_H,\mathbb{Z})^0$ has no nonzero elements invariant under monodromy, we see that $H^2(S_l,\mathbb{Z})^0$ lies in $H^2(Q_{l,H},\mathbb{Z})^0$. Since $H^2(Q_{l,H},\mathbb{Z})^0$ has no nonzero element invariant under monodromy, we have $H^2(S_l,\mathbb{Z})^0=0$ and $H^2(S_l,\mathbb{Q})=\mathbb{Q}h\oplus\mathbb{Q}\sigma_1$. $\square$

We will prove Theorem 5.8 in conjunction with some results on the cohomology of S_l and by induction as follows.

Theorem 5.13. 1. *The maps* $\psi^0 : H^{n-3}(S_l,\mathbb{Z})^0\to H^{n-1}(X,\mathbb{Z})^0$ *and* $\psi'^0 : H^{n-3}(S'_l,\mathbb{Z})^0\to H^{n-1}(X,\mathbb{Z})^0$ *are onto. The kernel of* ψ'^0 *is the* i_l*-invariant part* $H^{n-3}(S'_l,\mathbb{Z})^{0+}$ *of* $H^{n-3}(S'_l,\mathbb{Z})^0$ *and therefore the kernel of* ψ^0 *is* $H^{n-3}(S_l,\mathbb{Z})\cap H^{n-3}(S'_l,\mathbb{Z})^{0+}$*.*

2. *The cohomology of* S_l *is torsion in odd degree except in degree* $n-3$*.*
3. *In even degree the rational cohomology of* S_l *is generated by monomials in* h *and* σ_1 *except in degree* $n-3$*.*

Proof : As mentioned above, we proceed by induction on n.

We first show that, for any given $n\geq 5$, parts 2 and 3 of the theorem imply part 1.

Indeed, assume that parts 2 and 3 are true for any smooth cubic hypersurface in $\mathbb{P}^n$ for a fixed n. Let $Sym(h,\sigma_1)$ be the subvector space of $H^{n-3}(S_l,\mathbb{Q})$ generated by monomials in h and σ_1 ($Sym(h,\sigma_1)=0$ if n is even). Then, if n is odd, it follows from numbers 2 and 3 that we have the decomposition

$$H^{n-3}(S_l,\mathbb{Q})\cong H^{n-3}(S_l,\mathbb{Q})^0\oplus Sym(h,\sigma_1)\,.$$

Since $Sym(h,\sigma_1)$ is algebraic, its image in $H^{n-1}(X,\mathbb{Z})$ is also algebraic. For X generic $H^{n-1}(X,\mathbb{Z})^0$ has no algebraic part. Therefore for X generic and hence for all X, the image of $Sym(h,\sigma_1)$ is zero in $H^{n-1}(X,\mathbb{Z})^0$. Since the cohomology of X has no torsion and, by Theorem 5.1, the map $\psi : H^{n-3}(S_l,\mathbb{Z})\to H^{n-1}(X,\mathbb{Z})$ is onto, it follows that

$$\psi^0 : H^{n-3}(S_l,\mathbb{Z})^0\longrightarrow H^{n-1}(X,\mathbb{Z})^0$$

is onto.

Since ψ^0 is the composition of ψ'^0 with the inclusion $H^{n-3}(S_l,\mathbb{Z})^0\hookrightarrow H^{n-3}(S'_l,\mathbb{Z})^0$, we deduce that ψ'^0 is also onto. The rest of part 1 is Corollary 5.11.

Now we prove that parts 1, 2 and 3 for $n-1\geq 5$ imply parts 2 and 3 for n. Let $H_1,H_2,X_{l,H},S_{l,H}$ be as in the proof of Lemma 5.12, let H'_1 be the proper transform of H_1 in $\mathbb{P}^{n-2'}$ and let $X'_{l,H}$ and $S'_{l,H}$ be the proper transforms of $X_{l,H}$ and $S_{l,H}$ in X'_l and S'_l respectively. By [9] pages 23-25, for every $k\leq n-5$, we have

$$H^k(S_l,\mathbb{Z})\cong H^k(S_{l,H},\mathbb{Z})$$

and

$$H^{n-4}(S_l,\mathbb{Z}) \hookrightarrow H^{n-4}(S_{l,H},\mathbb{Z}) \ .$$

In particular, it follows from this and our induction hypothesis that $H^{n-3}(S_l,\mathbb{Q})$ and $H^{n-4}(S_l,\mathbb{Q})$ are the direct sums of their primitive parts and their subvector spaces generated by the monomials in h and σ_1. Now it is enough to show that $H^{n-4}(S_l,\mathbb{Q})^0 = 0$.

If we choose a general pencil of hyperplanes in $\mathbb{P}^{n-2}$ of which H_1 is a member, then $H^{n-4}(S_l,\mathbb{Z})$ maps into the part of $H^{n-4}(S_{l,H},\mathbb{Z})$ which is invariant under monodromy. By our induction hypothesis, we have the exact sequence

$$0 \longrightarrow H^{n-4}(S_{l,H},\mathbb{Z})^0 \cap H^{n-4}(S'_{l,H},\mathbb{Z})^{0+} \longrightarrow H^{n-4}(S_{l,H},\mathbb{Z})^0 \longrightarrow H^{n-2}(X_H,\mathbb{Z})^0 \longrightarrow 0 \ .$$

Since $H^{n-2}(X_H,\mathbb{Z})^0$ has no nonzero elements invariant under monodromy, we see that $H^{n-4}(S_l,\mathbb{Z})^0$ lies in $H^{n-4}(S_{l,H},\mathbb{Z})^0 \cap H^{n-4}(S'_{l,H},\mathbb{Z})^{0+}$. Therefore all the elements of $H^{n-4}(S_l,\mathbb{Z})^0$ are i_l-invariant and hence are contained in $H^{n-4}(Q'_l,\mathbb{Z})^0 \subset H^{n-4}(S'_l,\mathbb{Z})^0$.

Now let

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, phantom, "\subset"] \arrow[d] & \mathbb{P}^{n+1} \arrow[d] \\
\mathbb{P}^{n-2} \arrow[r, phantom, "\subset"] & \mathbb{P}^{n-1}
\end{tikzcd}$$

be a commutative diagram of linear embeddings and projections from l. Let Y be a general cubic hypersurface in $\mathbb{P}^{n+1}$ such that $Y \cap \mathbb{P}^n = X$, let Y_l be the blow up of Y along l and let $S_{l,Y}$ be the variety parametrizing lines in the fibers of $Y_l \to \mathbb{P}^{n-1}$. Then, again by [9] pages 23-25, we have

$$H^{n-4}(S_l,\mathbb{Z}) \cong H^{n-4}(S_{l,Y},\mathbb{Z}) \ .$$

Let $T_{l,Y}$ be the variety parametrizing the planes in the fibers of $Y_l \to \mathbb{P}^{n-1}$ and similarly define $Q_{l,Y}$, $Q'_{l,Y}$, $R'_{l,Y}$ and $Q''_{l,Y}$. By Lemma 5.5 we have the exact sequence

$$0 \longrightarrow H^{n-2}(T_{l,Y},\mathbb{Q}) \oplus H^{n-4}(T_{l,Y},\mathbb{Q}) \oplus H^{n-6}(T_{l,Y},\mathbb{Q})^{\oplus 2} \oplus H^{n-8}(T_{l,Y},\mathbb{Q}) \oplus H^{n-2}(R'_{l,Y},\mathbb{Q}) \longrightarrow$$

$$\longrightarrow H^{n}(Q''_{l,Y},\mathbb{Q}) \longrightarrow H^{n+2}(\mathbb{P}^{n-1},\mathbb{Q}) \longrightarrow 0 \ .$$

It is easily seen that the intersection of the subspace

$$H^{n-2}(T_{l,Y},\mathbb{Q}) \oplus H^{n-4}(T_{l,Y},\mathbb{Q}) \oplus H^{n-6}(T_{l,Y},\mathbb{Q})^{\oplus 2} \oplus H^{n-8}(T_{l,Y},\mathbb{Q}) \oplus H^{n-2}(R'_{l,Y},\mathbb{Q})$$

of $H^n(Q''_{l,Y},\mathbb{Q}) \supset H^n(Q'_{l,Y},\mathbb{Q})$ with $H^n(S_{l,Y},\mathbb{Q}) \subset H^n(S''_{l,Y},\mathbb{Q})$ is zero. It immediately follows that $H^{n-4}(S_{l,Y},\mathbb{Q})^0 = H^{n-4}(S_l,\mathbb{Q})^0 = 0$.

To finish the proof of the theorem all we need to do is to prove the theorem in the case $n = 5$. Suppose therefore that $n = 5$. Then part 3 is clear. Part 2 is proved in [15] Lemme 3 page 591. Part 1 is Lemma 5.12. $\square$

6. The proof of Theorem 4

Let $\beta : \mathcal{L} \to F$ be the family of lines in X with $\iota : \mathcal{L} \to X$ the natural morphism which is inclusion on each fiber of β. The map ϕ in Theorem 4 is the composition $H^{n-1}(X, \mathbb{Z})^0 \hookrightarrow H^{n-1}(X, \mathbb{Z}) \xrightarrow{\beta_*\iota^*} H^{n-3}(F, \mathbb{Z}) \twoheadrightarrow H^{n-3}(F, \mathbb{Z})^0$. To prove Theorem 4 consider the diagram (similar to diagram 11.7 on page 331 of [6])

$$\begin{tikzcd}
H^{n-1}(X,\mathbb{Z})^0 \arrow[r, "\phi"] & H^{n-3}(F,\mathbb{Z})^0 \arrow[r, "j^*"] & H^{n-3}(S'_l,\mathbb{Z})^0 \arrow[d, "t"'] \\
H_{n-1}(X,\mathbb{Z})^0 \arrow[u, "s"] & H_{n-3}(F,\mathbb{Z})^0 \arrow[l, "\chi"'] & H_{n-3}(S'_l,\mathbb{Z})^0 \arrow[l, "j_*"']
\end{tikzcd}$$

where the vertical arrows are induced by Poincaré Duality, the map $j : S'_l \to F$ is the composition of $S'_l \to S_l \to D_l$ with the inclusion $D_l \hookrightarrow F$, and χ (equal to the composition $H_{n-3}(F, \mathbb{Z})^0 \hookrightarrow H_{n-3}(F, \mathbb{Z}) \xrightarrow{\iota_*\beta^*} H_{n-1}(X, \mathbb{Z}) \twoheadrightarrow H_{n-1}(X, \mathbb{Z})^0$) is the transpose of ϕ. We prove that χ is an isomorphism. Since ψ'^0 ($= \chi j_*$ after identification of the cohomology groups of X and S'_l with homology groups by Poincaré Duality) is surjective, so is χ. It remains to prove that χ is also injective. For this we will prove that the composition $j_* t j^* \phi s \chi$ is equal to multiplication by -2. Let α be a topological cycle on F with homology class $[\alpha] \in H_{n-3}(F, \mathbb{Z})^0$. We can, and will, suppose that α is transverse to D_l. Then it is immediately seen that $j_* t j^* \phi s \chi([\alpha])$ is represented by the cycle parametrizing lines on X which are incident to l as well as to some line parametrized by α. Let l' be any line in X not incident to l. Then there are at most five lines in X incident to both l and l'. Suppose that there are five distinct lines $l_1, \dots, l_5$ in X intersecting each of l and l' in five distinct points. This condition will be satisfied by a general line l' in X. Let P_3 be the space spanned by l and l'. We have

Lemma 6.1. *There is exactly a pencil of cubic surfaces in P_3 containing l, l' and $l_1, \dots, l_5$. Furthermore, the cubic surfaces of this pencil are all tangent along l and l'.*

Proof : A dimension count shows that there is at least a pencil of cubic surfaces containing l, l' and $l_1, \dots, l_5$. Any two such cubic surfaces are tangent at five points along l. It is easily seen then that the two surfaces are tangent everywhere on l. Similarly, they are tangent everywhere on l'. This implies now that there is exactly a pencil of cubic surfaces containing l, l' and $l_1, \dots, l_5$. $\square$

Therefore, on X, the cycle $2[l] + 2[l'] + [l_1] + \ldots + [l_5]$ is a complete intersection of divisors. By continuity, this will be the case whenever l and l' do not intersect (even if some of the l_i "come together"). This is easily seen to imply that, in F, the sum of the cycle 2α with the cycle parametrizing lines incident to l and to some line of α is homologous to a multiple of a power of the hyperplane class on F. Hence the sum is zero in the primitive homology of F and $j_*tj^*\phi s\chi([\alpha]) = -2[\alpha]$. Therefore $j_*tj^*\phi s\chi$ is equal to multiplication by -2 as claimed. In particular, it is injective and so is χ. Hence χ is an isomorphism and so is its transpose ϕ.

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