

DENSITY AND COMPLETENESS OF SUBVARIETIES OF MODULI SPACES OF CURVES OR ABELIAN VARIETIES

E. IZADI

ABSTRACT. Let V be a subvariety of codimension $\leq g$ of the moduli space \mathcal{A}_g of principally polarized abelian varieties of dimension g or of the moduli space $\widetilde{\mathcal{M}}_g$ of curves of compact type of genus g over \mathbb{C} . We prove that the set $E_1(V)$ of elements of V which map onto an elliptic curve is analytically dense in V . From this we deduce that if $V \subset \mathcal{A}_g$ is complete, then V has codimension equal to g and the set of elements of V isogenous to a product of g elliptic curves is countable and analytically dense in V . We also prove a technical property of the conormal sheaf of V if $V \subset \widetilde{\mathcal{M}}_g$ (or \mathcal{A}_g) is complete of codimension g .

INTRODUCTION

Let \mathcal{M}_g be the moduli space of smooth curves of genus $g \geq 2$ and let \mathcal{A}_g be the moduli space of principally polarized abelian varieties (*ppav*) of dimension g over \mathbb{C} . A (Deligne-Mumford) stable curve of genus g is a reduced, connected and complete curve of arithmetic genus g with only nodes as singularities and with finite automorphism group. We say that a stable curve is of compact type if its generalized jacobian is an abelian variety. We denote by $\widetilde{\mathcal{M}}_g$ the moduli space of stable curves of compact type and genus g over \mathbb{C} . By “density” we always mean “analytic density” unless we specify otherwise.

Given a subvariety V of \mathcal{M}_g or $\widetilde{\mathcal{M}}_g$ and an integer q between 1 and $g/2$, let $E_q(V)$ be the subset of V parametrizing curves whose jacobian contains an abelian variety of dimension q . We define $E_q(V)$ for V a subvariety of \mathcal{A}_g in a similar fashion. It is well-known that $E_q(\mathcal{A}_g)$ is dense in \mathcal{A}_g for all q . Colombo and Pirola pose the following question in [3]

Problem 1. *When is $E_q(V)$ dense in V ?*

Colombo and Pirola give a sufficient condition for the density of $E_q(V)$ in V . They then show that $E_1(V)$ is dense in V for all subvarieties V of \mathcal{M}_g of codimension at most $g - 1$. They deduce

Date: July 1996.

The author was partially supported by an NSF grant.

from this a second proof of the noncompleteness of codimension $g - 1$ subvarieties of $\widetilde{\mathcal{M}}_g$ which was originally proved by Diaz in [5], Corollary page 80 (Colombo and Pirola prove the noncompleteness of codimension $g - 1$ subvarieties of $\widetilde{\mathcal{M}}_g$ which meet \mathcal{M}_g ; however, if the subvariety is contained in $\widetilde{\mathcal{M}}_g \setminus \mathcal{M}_g$, its noncompleteness can be easily seen by mapping it (or, in some cases, a double cover of it) to a moduli space of curves of lower genus).

Using the condition of Colombo and Pirola, we show

Theorem 1. *Suppose that $g \geq 2$. Let V be a subvariety of codimension at most g of \mathcal{M}_g or \mathcal{A}_g , then $E_1(V)$ is dense in V .*

This result brings out another fundamental difference between the moduli spaces in characteristic zero and in positive characteristic (see Section 2 below).

We also obtain

Corollary 1. *Suppose that $g \geq 2$. Let V be a complete subvariety of codimension at most g of \mathcal{A}_g . For all $i \in \{1, \dots, g\}$, denote by $E_{1,i}(V)$ the subset of V parametrizing ppav's isogenous to a product of a ppav of dimension $g - i$ and i elliptic curves. Then*

1. *the variety V has codimension exactly g ,*
2. *for all $i \in \{1, \dots, g\}$, any irreducible component Z of $E_{1,i}(V)$ has the expected dimension $\frac{(g-i)(g-i+1)}{2} + i - g$; furthermore, the variety Z parametrizes ppav's isogenous to a product of i fixed elliptic curves (depending only on Z) and some ppav of dimension $g - i$,*
3. *for all $q, 1 \leq q \leq g/2$, any irreducible component of $E_q(V)$ has the expected dimension $\frac{(g-q)(g-q+1)}{2} + \frac{q(q+1)}{2} - g$,*
4. *the set $E_{1,g}(V)$ is dense in V . In particular, the set $E_{1,g}(V)$ is (countable) infinite and, for all $i \in \{1, \dots, g\}$, for all $q \in \{1, \dots, [g/2]\}$, the sets $E_{1,i}(V)$ and $E_q(V)$ are dense in V (since they contain $E_{1,g}(V)$).*

An immediate consequence of Corollary 1 is that complete subvarieties of \mathcal{A}_g have codimension at least g . There are different proofs of this last fact in [12] (2.5.1 page 231) and [7] (Corollary 1.7). There are no known examples of complete subvarieties of codimension g of \mathcal{A}_g (or $\widetilde{\mathcal{M}}_g$) except for $g = 2$, although g is the best known lower bound for the codimension of complete subvarieties of \mathcal{A}_g (and $\widetilde{\mathcal{M}}_g$). It is conjectured in [12] (2.3 page 230) that for $g \geq 3$ the codimension of a complete subvariety of \mathcal{A}_g is at least $g + 1$.

Let \mathcal{H}_g be the locus of hyperelliptic curves in \mathcal{M}_g . Let $\widetilde{\mathcal{M}}'_g$ and \mathcal{A}'_g be respectively the moduli space of curves of compact type with level n structure (for some fixed $n \geq 3$) and the moduli space of ppav's with level n structure. Let $s_a : \mathcal{A}'_g \rightarrow \mathcal{A}_g$ and $s_c : \widetilde{\mathcal{M}}'_g \rightarrow \widetilde{\mathcal{M}}_g$ be the natural morphisms. It is well-known that \mathcal{A}'_g and $\mathcal{M}'_g := s_c^{-1}(\mathcal{M}_g)$ are smooth and that there is a universal family of abelian varieties with level n structure on \mathcal{A}'_g and a universal family of (smooth) curves with level n structure on \mathcal{M}'_g . By a “universal” family of curves or abelian varieties with level n structure we mean a family which solves the moduli problem for curves or abelian varieties with level n structure. We note that the only properties of $\widetilde{\mathcal{M}}'_g$, \mathcal{M}'_g and \mathcal{A}'_g we need are the smoothness of \mathcal{M}'_g and \mathcal{A}'_g and the existence of the universal families (also note that with non-abelian level structures or Prym-level structures, one can get smooth covers of $\widetilde{\mathcal{M}}'_g$ as well (see [8] and [14])). We have the following technical consequence of our results.

Corollary 2. *Suppose that $g \geq 3$. Let V be a complete subvariety of codimension g of $\widetilde{\mathcal{M}}_g$ or \mathcal{A}_g . If $V \subset \mathcal{A}_g$, let V_0 be the smooth locus of $s_a^{-1}(V)$. If $V \subset \widetilde{\mathcal{M}}_g$, let V_0 be the smooth locus of $s_c^{-1}(V \cap (\mathcal{M}_g \setminus \mathcal{H}_g))$. Then the conormal bundle to V_0 is isomorphic to the tensor product of the Hodge bundle (the pushforward of the sheaf of relative one-forms on the universal abelian (or jacobian) variety) with a subline bundle of the Hodge bundle.*

Finally, we point out that, if $\mathcal{A}_{g,d}$ denotes the moduli space of abelian varieties of dimension g and polarization type $d = (d_1, \dots, d_g)$, then there is a finite correspondence between \mathcal{A}_g and $\mathcal{A}_{g,d}$ so that our results remain valid if we replace \mathcal{A}_g with $\mathcal{A}_{g,d}$.

ACKNOWLEDGMENTS

I wish to thank F. Oort for mentioning these questions to me and also for stimulating discussions.

NOTATION

For any vector space or vector bundle (resp. affine cone) E , we denote by $\mathbb{P}(E)$ the projective space (resp. projective variety) of lines in E and by E^* its dual vector space or vector bundle. We let $E^{\otimes m}$, $S^m E$ and $\Lambda^m E$ respectively be the m -th tensor power, the m -th symmetric power and the m -th alternating power of E . For any linear map of vector spaces or vector bundles $l : E \rightarrow F$, we denote by $\bar{l} : \mathbb{P}(E) \rightarrow \mathbb{P}(F)$ its projectivization.

For any variety X and any point $x \in X$, we denote by $T_x X$ the Zariski tangent space to X at x and by $T_x^* X$ the dual of $T_x X$. We denote by X_{sm} the subvariety of smooth points of X .

For a ppav A , we let $\rho : H^0(\Omega_A^1)^{\otimes 2} \longrightarrow S^2 H^0(\Omega_A^1)$ be the natural linear map with kernel $\Lambda^2 H^0(\Omega_A^1)$. For a subvariety V of \mathcal{A}'_g the restriction to V of the universal family on \mathcal{A}'_g gives a family \mathcal{A}_V of ppav's on V (we forget the level n structure). For a point t of V , we let A_t be the fiber of \mathcal{A}_V at t . The Zariski-tangent space to \mathcal{A}'_g at t can be canonically identified with $S^2 H^0(\Omega_{A_t}^1)^*$. We denote by $\pi_a : S^2 H^0(\Omega_{A_t}^1) \longrightarrow T_t^* V$ the codifferential at t of the embedding $V \hookrightarrow \mathcal{A}'_g$.

For a smooth curve C , we denote by ω_C the canonical sheaf of C and let κC be the image of C in the dual projective space $|\omega_C|^*$ of the linear system $|\omega_C|$ by the natural morphism associated to this linear system. If $A = JC$ is the jacobian of a smooth curve C , then $H^0(\Omega_A^1) \cong H^0(\omega_C)$. Let $m : S^2 H^0(\omega_C) \longrightarrow H^0(\omega_C^{\otimes 2})$ be multiplication and put $\mu := m\rho$. For $V \subset \mathcal{M}'_g$ the restriction to V of the universal family on \mathcal{M}'_g gives a family \mathcal{C}_V of curves on V (again we forget the level n structure). For $t \in V$ we let C_t be the fiber of \mathcal{C}_V at t . The Zariski-tangent space to \mathcal{M}'_g at t can be canonically identified with $H^0(\omega_{C_t}^{\otimes 2})^*$. We let $\pi : H^0(\omega_{C_t}^{\otimes 2}) \longrightarrow T_t^* V$ be the codifferential at t of the embedding $V \hookrightarrow \mathcal{M}'_g$.

For a stable curve C of compact type (resp. a ppav A), we will call the corresponding point of $\widetilde{\mathcal{M}}_g$ (resp. \mathcal{A}_g) the moduli point of C (resp. A).

1. THE PROOFS

In this section we give the proof of Theorem 1 and its corollaries.

We first consider the case where V is contained in \mathcal{M}_g . We may and will replace V with its inverse image in \mathcal{M}'_g .

The relative jacobian of \mathcal{C}_V gives us a family of ppav's on V . We can therefore apply Theorem (1) on page 162 of [3]: to show that $E_1(V)$ is dense in V it is enough to prove the following:

There exists a Zariski dense (Zariski-)open subset U of V , contained in the smooth locus V_{sm} of V , such that, for all $t \in U$, there is a subvector space W of $H^0(\omega_{C_t})$ which has dimension 1 and is such that the composition

$$W \otimes W^\perp \xrightarrow{\mu} H^0(\omega_{C_t}^{\otimes 2}) \xrightarrow{\pi} T_t^* V$$

is injective. Here W^\perp is the orthogonal complement of W with respect to the hermitian form on $H^0(\omega_{C_t})$ induced by the natural polarization of JC_t . We sketch briefly how this condition is obtained in the more general case where W has dimension q with $1 \leq q \leq g/2$ and $V \subset \mathcal{A}'_g$ has any dimension $\geq q(g - q)$.

To prove the density of $E_q(V)$ in V , it is enough to show that there is a Zariski dense open subset U of V_{sm} , such that, for all $t \in U$, there is an analytic neighborhood U' of t , $U' \subset U$, such that $E_q(V) \cap U'$ is dense in U' . An abelian variety A contains an abelian subvariety of dimension q if and only if $H^0(\Omega_A^1)$ contains a q -dimensional \mathbb{C} -subvector space which is the tensor product with \mathbb{R} of a vector subspace of dimension $2q$ of $H^1(A, \mathbb{Q})$ (after identifying $H^0(\Omega_A^1)$ with $H^1(A, \mathbb{R}) \cong H^1(A, \mathbb{Q}) \otimes \mathbb{R}$ as real vector spaces). Let t be an element of V_{sm} . For a contractible analytically open set $U' \ni t$ contained in V_{sm} , let $F_{U'}$ be the Hodge bundle over U' . Then one can trivialize $F_{U'}$ as a real vector bundle using the Gauss-Manin connection. Therefore the grassmannian bundle of $2q$ -dimensional real subvector spaces of $F_{U'}$ is isomorphic to $U' \times G_{\mathbb{R}}(2q, 2g)$, where $G_{\mathbb{R}}(2q, 2g)$ is the Grassmannian of $2q$ -dimensional \mathbb{R} -subvector spaces of $H^1(A_t, \mathbb{R})$. Hence there is a well-defined map $\Phi : G(q, F_{U'}) \longrightarrow G_{\mathbb{R}}(2q, 2g)$ where $G(q, F_{U'})$ is the Grassmannian of q -dimensional \mathbb{C} -subvector spaces of $F_{U'}$: The map Φ sends a q -dimensional complex subvector space of $H^0(\Omega_{A_s}^1)$ (with $s \in U'$) to the image of its underlying real vector space under the isomorphism $H^1(A_s, \mathbb{R}) \xrightarrow{\cong} H^1(A_t, \mathbb{R})$ obtained from the \mathbb{R} -trivialization of $F_{U'}$. Let $G_{\mathbb{Q}}(2q, 2g) \subset G_{\mathbb{R}}(2q, 2g)$ be the Grassmannian of $2q$ -dimensional \mathbb{Q} -subvector spaces of $H^1(A_t, \mathbb{Q})$ and let $p : G(q, F_{U'}) \longrightarrow U'$ be the natural morphism. Then $E_q(V) \cap U' = p(\Phi^{-1}(G_{\mathbb{Q}}(2q, 2g)))$. To prove the density of $E_q(V) \cap U'$ in U' , it is enough to prove that there is a subset \mathcal{Y} of $G(q, F_{U'})$ such that $p(\mathcal{Y}) = U'$ and $\Phi^{-1}(G_{\mathbb{Q}}(2q, 2g)) \cap \mathcal{Y}$ is dense in \mathcal{Y} . Since $G_{\mathbb{Q}}(2q, 2g)$ is dense in $G_{\mathbb{R}}(2q, 2g)$, it is enough to find \mathcal{Y} such that $p(\mathcal{Y}) = U'$ and $\Phi|_{\mathcal{Y}}$ is an open map. If Φ has maximal rank (i.e., the differential $d\Phi$ of Φ is surjective) everywhere on \mathcal{Y} , then $\Phi|_{\mathcal{Y}}$ is an open map. Therefore $E_q(V) \cap U'$ is dense in U' if for every $s \in U'$ there is a q -dimensional \mathbb{C} -subvector space W of $H^0(\Omega_{A_s}^1)$ such that $d\Phi$ is surjective at $(W, s) \in G(q, F_{U'})$ (then \mathcal{Y} would be the set of such (W, s)). The tangent space $T_{(W,s)}G(q, F_{U'})$ is isomorphic to $W \otimes W^\perp \oplus T_s U'$, the tangent space to $G_{\mathbb{R}}(2q, 2g)$ at $\Phi(W, s)$ is isomorphic to $W \otimes W^\perp \oplus \overline{W \otimes W^\perp} \cong W \otimes W^\perp \oplus (W \otimes W^\perp)^*$ and the restriction of $d\Phi$ to the $W \otimes W^\perp$ summand of $T_{(W,s)}G(q, F_{U'})$ is an isomorphism onto the $W \otimes W^\perp$ summand of $T_{\Phi(W,s)}G_{\mathbb{R}}(2q, 2g)$. Therefore $d\Phi$ is surjective if and only if the map it induces $T_s U' = \frac{T_{(W,s)}G(q, F_{U'})}{W \otimes W^\perp} \longrightarrow (W \otimes W^\perp)^* = \frac{T_{\Phi(W,s)}G_{\mathbb{R}}(2q, 2g)}{W \otimes W^\perp}$ is surjective, i.e., if and only if the dualized map $W \otimes W^\perp \longrightarrow T_s^* U'$ is injective. Let F be the Hodge bundle over the Siegel upper half space \mathcal{U}_g . The inclusion $U' \hookrightarrow \mathcal{A}'_g$ lifts to an inclusion $U' \hookrightarrow \mathcal{U}_g$ because U' is contractible and there is a family of ppav's on U' (the restriction of \mathcal{A}_V). Factoring Φ through the Grassmannian of q -planes in F over \mathcal{U}_g , the map $W \otimes W^\perp \longrightarrow T_s^* U'$ can be seen to be the composition

$$W \otimes W^\perp \xrightarrow{\rho} S^2 H^0(\Omega_{A_s}^1) \xrightarrow{\pi_a} T_s^* U' = T_s^* V.$$

For V contained in \mathcal{M}'_g , we have $\pi_a = \pi m$.

Clearly, if $\pi\mu : W \otimes H^0(\omega_{C_t}) \longrightarrow T_t^*V$ is injective, then so is $\pi\mu : W \otimes W^\perp \longrightarrow T_t^*V$. In view of this (and also for use in the proof of Corollary 2) we show:

Proposition 1.1. *Suppose that $g \geq 3$. Let V be a subvariety of codimension at most g of \mathcal{M}'_g . Let t be a point of V_{sm} and let N be the kernel of $\pi : H^0(\omega_{C_t}^{\otimes 2}) \longrightarrow T_t^*V$.*

1. *Suppose that C_t is non-hyperelliptic. Suppose that, for any one-dimensional subvector space W of $H^0(\omega_{C_t})$, the map $\pi\mu : W \otimes H^0(\omega_{C_t}) \longrightarrow T_t^*V$ is not injective. Then V has codimension exactly g and there is a one-dimensional subvector space W_N of $H^0(\omega_{C_t})$ such that $N = \mu(W_N \otimes H^0(\omega_{C_t}))$.*
2. *Suppose that C_t is hyperelliptic and that V is not transverse to $\mathcal{H}'_g := s_c^{-1}(\mathcal{H}_g)$ at t (i.e., the sum $T_tV + T_t\mathcal{H}'_g \subset T_t\mathcal{M}'_g$ is not equal to $T_t\mathcal{M}'_g$). Then there exists a one-dimensional subvector space W of $H^0(\omega_{C_t})$ such that the map $\pi\mu : W \otimes H^0(\omega_{C_t}) \longrightarrow T_t^*V$ is injective.*

Proof: Consider the composition

$$\mathbb{P}(H^0(\omega_{C_t})^{\otimes 2}) \xrightarrow{\bar{\rho}} \mathbb{P}(S^2H^0(\omega_{C_t})) \xrightarrow{\bar{m}} \mathbb{P}(H^0(\omega_{C_t}^{\otimes 2})).$$

The kernel of m is the space $I_2(C_t)$ of quadratic forms vanishing on κC_t . Hence the *rational* map \bar{m} is the projection with center $\mathbb{P}(I_2(C_t))$. Let $\bar{\mathcal{S}}$ be the image by $\bar{\rho}$ of the Segre embedding \mathcal{S} of $\mathbb{P}(H^0(\omega_{C_t})) \times \mathbb{P}(H^0(\omega_{C_t}))$ in $\mathbb{P}(H^0(\omega_{C_t})^{\otimes 2})$. Let N' be the set of rank 2 symmetric tensors in $S^2H^0(\omega_{C_t})$ which lie in $m^{-1}(N)$ (then $\mathbb{P}(N')$ is the *reduced* intersection of $\bar{\mathcal{S}}$ and $\bar{m}^{-1}(\mathbb{P}(N))$).

Suppose that for all $W \subset H^0(\omega_{C_t})$ of dimension 1, the map $\mu : W \otimes H^0(\omega_{C_t}) \longrightarrow T_t^*V$ is not injective, i.e., for all $w \in H^0(\omega_{C_t})$, there is $w' \in H^0(\omega_{C_t})$ such that $\mu(w \otimes w') \in N$. This implies that the dimension of $\mathbb{P}(N')$ is at least $g - 1$. We will show below that this does not happen if C_t is hyperelliptic and V is not transverse to \mathcal{H}'_g at t . If C_t is non-hyperelliptic, we will show that this implies that $\mathbb{P}(N')$ is a linear subspace of $\bar{\mathcal{S}}$ and that its inverse image in \mathcal{S} is the union of two linear subspaces of \mathcal{S} which are two fibers of the two projections of \mathcal{S} onto \mathbb{P}^{g-1} and are exchanged under the involution of \mathcal{S} which interchanges the two \mathbb{P}^{g-1} -factors of \mathcal{S} . The proposition will then easily follow from this.

Suppose first that C_t is *non*-hyperelliptic. Then $m : S^2H^0(\omega_{C_t}) \longrightarrow H^0(\omega_{C_t}^{\otimes 2})$ is onto (see [2] page 117). We have

Lemma 1.2. *Suppose $g = 2$ or $g \geq 3$ and C_t is non-hyperelliptic. Suppose that for all $W \subset H^0(\omega_{C_t})$ of dimension 1, the map $\mu : W \otimes H^0(\omega_{C_t}) \longrightarrow T_t^*V$ is not injective. Then the map $\mathbb{P}(N') \longrightarrow \mathbb{P}(N)$ is generically one-to-one.*

Proof : If not, then, for all $w \in H^0(\omega_{C_t})$, there exists $w', w_1, w'_1 \in H^0(\omega_{C_t})$ such that $ww' := \rho(w \otimes w')$ and $w_1w'_1 := \rho(w_1 \otimes w'_1)$ are not proportional but $m(ww')$ and $m(w_1w'_1)$ are proportional elements of N . Therefore, supposing w general, there exists $\lambda \in \mathbb{C}, \lambda \neq 0$, such that the element $\lambda ww' - w_1w'_1$ of $S^2H^0(\omega_{C_t})$ lies in $I_2(C_t)$, i.e., defines a quadric $q(w)$ of rank 3 or 4 (in the canonical space $|\omega_{C_t}|^*$) which contains κC_t (the canonical curve κC_t is not contained in any quadric of rank ≤ 2 since it is nondegenerate). If $g \leq 3$, this is impossible because in that case $I_2(C_t) = 0$. If $g \geq 4$, the intersection L of the two hyperplanes in $|\omega_{C_t}|^*$ with equations w and w_1 is an element of a ruling of the quadric $q(w)$. Therefore L cuts a divisor of a g_d^1 (a g_d^1 is a pencil of divisors of degree d) on C_t with $d \leq g - 1$ (see [1], Lemmas 2 and 3 page 192). Therefore the divisor of zeros of w on C_t contains a divisor of a g_d^1 . By the uniform position Theorem (see [2] Chapter 3, §1) this does not happen for w in some nonempty Zariski-open subset of $H^0(\omega_{C_t}) \setminus \{0\}$. \square

Therefore, since the dimension of $\mathbb{P}(N')$ is at least $g - 1$ and the dimension of $\mathbb{P}(N)$ is at most $g - 1$, the map $\mathbb{P}(N') \longrightarrow \mathbb{P}(N)$ is birational and $\mathbb{P}(N')$ and $\mathbb{P}(N)$ have both dimension $g - 1$.

This proves, in particular, that V has codimension exactly g .

Since no quadrics of rank ≤ 2 contain κC_t , the center $\mathbb{P}I_2(C_t)$ of the projection \overline{m} does not intersect $\overline{\mathcal{S}}$. In particular, the space $\mathbb{P}I_2(C_t)$ does not intersect $\mathbb{P}(N')$. Therefore \overline{m} restricts to a birational *morphism* $\mathbb{P}(N') \longrightarrow \mathbb{P}(N)$ and, since $\mathbb{P}(N)$ is a linear subspace of $\mathbb{P}(H^0(\omega_{C_t}^{\otimes 2}))$, the degree of $\mathbb{P}(N')$ (in the projective space $\mathbb{P}(S^2H^0(\omega_{C_t}))$) is equal to the (generic) degree of the map $\mathbb{P}(N') \longrightarrow \mathbb{P}(N)$. Hence $\mathbb{P}(N')$ is a linear subspace of $\mathbb{P}(S^2H^0(\omega_{C_t}))$ and \overline{m} restricts to an *isomorphism* $\mathbb{P}(N') \xrightarrow{\cong} \mathbb{P}(N)$.

Let N'' be the cone of decomposable tensors in $H^0(\omega_{C_t})^{\otimes 2}$ which lie in $\mu^{-1}(N)$ (then $\mathbb{P}(N'')$ is the *reduced* inverse image of $\mathbb{P}(N')$ in $\mathcal{S} \subset \mathbb{P}(H^0(\omega_{C_t})^{\otimes 2})$). The map $\mathcal{S} \longrightarrow \overline{\mathcal{S}}$ is a finite *morphism* of degree 2 ramified on the diagonal. Therefore the map $\mathbb{P}(N'') \longrightarrow \mathbb{P}(N')$ is a morphism of degree ≤ 2 . Since the diagonal of $\overline{\mathcal{S}} \cong S^2\mathbb{P}(H^0(\omega_{C_t}))$ is irreducible of dimension $g - 1$ and spans $\mathbb{P}(S^2H^0(\omega_{C_t}))$, the space $\mathbb{P}(N')$ intersects this diagonal in a subvariety of dimension at most $g - 2$. Therefore the morphism $\mathbb{P}(N'') \longrightarrow \mathbb{P}(N')$ has degree 2 and $\mathbb{P}(N'')$ has degree 2 in $\mathbb{P}(H^0(\omega_{C_t})^{\otimes 2})$.

If $\mathbb{P}(N'')$ is irreducible, it spans a linear subspace $\tilde{\mathbb{P}}$ of $\mathbb{P}(H^0(\omega_{C_t})^{\otimes 2})$ of dimension g . This implies that $\mathbb{P}(\Lambda^2 H^0(\omega_{C_t}))$ intersects $\tilde{\mathbb{P}}$ in exactly one point. For $w_1, w_2 \in H^0(\omega_{C_t})$, let w'_1, w'_2 be such that $\mu(w_1 \otimes w'_1), \mu(w_2 \otimes w'_2) \in N$. For w_i general, w'_i is not proportional to w_i since $\mathbb{P}(N'')$ intersects the diagonal of \mathcal{S} in a subvariety of dimension at most $g - 2$. Therefore the lines spanned by $w_1 \otimes w'_1 - w'_1 \otimes w_1$ and $w_2 \otimes w'_2 - w'_2 \otimes w_2$ give us elements of $\tilde{\mathbb{P}} \cap \mathbb{P}(\Lambda^2 H^0(\omega_{C_t}))$ which is a point. Therefore, for all $w_1, w_2 \in H^0(\omega_{C_t})$ general there exists $\lambda \in \mathbb{C}, \lambda \neq 0$, such that

$$w_1 \otimes w'_1 - w'_1 \otimes w_1 = \lambda(w_2 \otimes w'_2 - w'_2 \otimes w_2)$$

Complete $\{w_1, w_2\}$ to a general basis $\{w_1, w_2, w_3, \dots, w_g\}$ of $H^0(\omega_{C_t})$ and write $w'_i = \sum_{1 \leq j \leq g} a_{ij} w_j$ for $i = 1$ or 2 . Then from the equation above we deduce $a_{1j} = a_{2j} = 0$ for $j > 2$. Therefore, w'_1 belongs to the span of w_1 and w_2 . Repeating this argument with w_1 and w_3 instead of w_1 and w_2 , we see that w'_1 also belongs to the span of w_1 and w_3 . Hence w'_1 is proportional to w_1 (this is the only part in the proof of Proposition 1.1 where we need $g \geq 3$). Contradiction.

Therefore $\mathbb{P}(N'')$ is reducible, i.e., it is the union of two linear subspaces of dimension $g - 1$. We have

Lemma 1.3. *Suppose $g \geq 2$. All linear subspaces of dimension $g - 1$ of $\mathcal{S} \subset \mathbb{P}(H^0(\omega_{C_t})^{\otimes 2}) \cong \mathbb{P}^{g^2-1}$ are elements of one of the two rulings of $\mathcal{S} \cong \mathbb{P}(H^0(\omega_{C_t})) \times \mathbb{P}(H^0(\omega_{C_t})) \cong \mathbb{P}^{g-1} \times \mathbb{P}^{g-1}$.*

Proof: Let T be a linear subspace of dimension $g - 1$ of \mathcal{S} . Let p_1 and p_2 be the two projections of $\mathcal{S} \cong \mathbb{P}^{g-1} \times \mathbb{P}^{g-1}$ onto its two factors. Let H_i be a general element of $p_i^*|\mathcal{O}_{\mathbb{P}^{g-1}}(1)|$ for $i = 1$ or 2 . Then $H_1 \cap T \neq H_2 \cap T$ and H_i does not contain T . In particular, the intersection $H_i \cap T$ is either empty or of dimension $g - 2$. The divisor $H_1 \cup H_2$ is the intersection of a hyperplane H in \mathbb{P}^{g^2-1} with \mathcal{S} . Since T is not contained in H_1 nor H_2 , the hyperplane H does not contain T and hence $T \cap H$ is a linear space of dimension $g - 2$. Since the two intersections $T \cap H_1 \neq T \cap H_2$ are both contained in the $(g - 2)$ -dimensional linear space $T \cap H$ and are either empty or have dimension $g - 2$, we have either $H_1 \cap T = \emptyset$ or $H_2 \cap T = \emptyset$. Suppose, for instance, that $H_1 \cap T = \emptyset$. It is easily seen that $p_1^{-1}(p_1(H_1)) = H_1$ implies $p_1(H_1) \cap p_1(T) = p_1(H_1 \cap T)$. Therefore $p_1(T)$ does not intersect $p_1(H_1)$ which is a hyperplane in \mathbb{P}^{g-1} . Hence $p_1(T)$ is a point and T is a fiber of p_1 . \square

We deduce from the above Lemma that $\mathbb{P}(N'') = \mathbb{P}(N_1) \cup \mathbb{P}(N_2)$ where $\mathbb{P}(N_1)$ and $\mathbb{P}(N_2)$ are elements of the two rulings of $\mathcal{S} \cong \mathbb{P}^{g-1} \times \mathbb{P}^{g-1}$. The spaces $\mathbb{P}(N_1)$ and $\mathbb{P}(N_2)$ are exchanged by the involution which exchanges the two factors of \mathcal{S} because $\mathbb{P}(N'')$ is the inverse image of a linear subspace in $\overline{\mathcal{S}} \cong S^2 \mathbb{P}^{g-1}$. Therefore there exists a one-dimensional subvector space W_N of $H^0(\omega_{C_t})$ such

that, for instance, $N_1 = W_N \otimes H^0(\omega_{C_t})$ and $N_2 = H^0(\omega_{C_t}) \otimes W_N$. So $N = \mu(N_1) = \mu(W_N \otimes H^0(\omega_{C_t}))$. This proves the Proposition in the non-hyperelliptic case.

Now suppose that C_t is hyperelliptic and that V is not transverse to \mathcal{H}'_g at t , i.e., the subspaces $T_t V$ and $T_t \mathcal{H}'_g$ do *not* span $T_t \mathcal{M}'_g$. Let ι be the hyperelliptic involution of C_t . Let $H^0(\omega_{C_t}^{\otimes 2})^+$ and $H^0(\omega_{C_t}^{\otimes 2})^-$ be the subvector spaces of $H^0(\omega_{C_t}^{\otimes 2})$ of ι -invariant and ι -anti-invariant quadratic differentials respectively. Then $H^0(\omega_{C_t}^{\otimes 2})^+$ is the image of $S^2 H^0(\omega_{C_t})$ by m and the conormal space to \mathcal{H}'_g at t can be canonically identified with $H^0(\omega_{C_t}^{\otimes 2})^-$. The non-transversality of V and \mathcal{H}'_g means that $N \cap H^0(\omega_{C_t}^{\otimes 2})^- \neq \{0\}$. This implies that N is not contained in $H^0(\omega_{C_t}^{\otimes 2})^+$. Since N has dimension at most g , the dimension of $N \cap H^0(\omega_{C_t}^{\otimes 2})^-$ is at most $g - 1$. Hence the dimension of $\mathbb{P}(N \cap H^0(\omega_{C_t}^{\otimes 2})^+) = \mathbb{P}(N) \cap \mathbb{P}(H^0(\omega_{C_t}^{\otimes 2})^+) = \mathbb{P}(N) \cap \overline{m}(\mathbb{P}(S^2 H^0(\omega_{C_t})))$ is at most $g - 2$. We have

Lemma 1.4. *Suppose $g \geq 2$ and C_t hyperelliptic. The map $\overline{m} : \overline{\mathcal{S}} \longrightarrow \overline{\overline{\mathcal{S}}} := \overline{m}(\overline{\mathcal{S}})$ is a finite morphism of degree $\frac{1}{2} \binom{2g-2}{g-1}$.*

Note that the lemma finishes the proof of Proposition 1.1: we saw above that the dimension of $\mathbb{P}(N) \cap \overline{m}(\mathbb{P}(S^2 H^0(\omega_{C_t})))$ is at most $g - 2$. A fortiori, since $\overline{m}(\mathbb{P}(S^2 H^0(\omega_{C_t}))) \supset \overline{\overline{\mathcal{S}}}$, the dimension of $\mathbb{P}(N) \cap \overline{\overline{\mathcal{S}}}$ is at most $g - 2$ and the dimension of $\mathbb{P}(N')$ is at most $g - 2$ which is what we needed to show (see the paragraphs preceding Lemma 1.2).

Proof of lemma 1.4: The map $\overline{m} : \overline{\mathcal{S}} \longrightarrow \overline{\overline{\mathcal{S}}}$ is a morphism if and only if the center $\mathbb{P}(I_2(C_t))$ of the projection \overline{m} does not intersect $\overline{\overline{\mathcal{S}}}$. This is the case because the canonical curve κC_t is nondegenerate and hence not contained in any quadrics of rank ≤ 2 .

Fix a nonzero element $ww' = \rho(w \otimes w')$ of $S^2 H^0(\omega_{C_t})$ and suppose that $w_1 w'_1 \in S^2 H^0(\omega_{C_t})$ is not proportional to ww' and $m(w_1 w'_1) = \lambda m(ww')$ for some $\lambda \in \mathbb{C}, \lambda \neq 0$. This is equivalent to $Z(w) + Z(w') = Z(w_1) + Z(w'_1)$ where $Z(w)$, for instance, is the divisor of zeros of w on the rational normal curve κC_t . So there are only a finite number of possibilities for $Z(w_1)$ and $Z(w'_1)$. This proves that $\overline{m} : \overline{\mathcal{S}} \longrightarrow \overline{\overline{\mathcal{S}}}$ is quasi-finite and hence finite since it is proper. Any divisor of degree $g - 1$ on $\kappa C_t \cong \mathbb{P}^1$ is the divisor of zeros of some element of $H^0(\omega_{C_t}) = H^0(\mathcal{O}_{\mathbb{P}^1}(g - 1))$, hence,

since there are $\frac{1}{2} \binom{2g-2}{g-1}$ ways to write a fixed reduced divisor of degree $2g-2$ as a sum of two divisors of degree $g-1$, the degree of $\overline{m} : \overline{\mathcal{S}} \rightarrow \overline{\overline{\mathcal{S}}}$ is $\frac{1}{2} \binom{2g-2}{g-1}$. \square

Proof of Theorem 1 in the case of curves: As explained in the beginning of this section, we need to find a Zariski-dense open subset U of V_{sm} , such that, for all $t \in U$, there exists $W \subset H^0(\omega_{C_t})$ (W of dimension 1) such that $\mu(W \otimes W^\perp) \cap N = \{0\}$.

First suppose $g \geq 3$. We may assume that V is irreducible. If V is contained in \mathcal{H}'_g , then V is not transverse anywhere to \mathcal{H}'_g and hence, by Proposition 1.1, we may take U to be all of V_{sm} . If $V \not\subset \mathcal{H}'_g$, take $U = V_{sm} \setminus \mathcal{H}'_g$. Suppose that there exists $t \in U$ such that, for all $W \subset H^0(\omega_{C_t})$ of dimension 1, we have $\mu(W \otimes W^\perp) \cap N \neq \{0\}$. Then, a fortiori, the hypotheses of part 1 of Proposition 1.1 are met and $N = \mu(W_N \otimes H^0(\omega_{C_t}))$. Then every element of $H^0(\omega_{C_t})$ is orthogonal to W_N . This is impossible given that the hermitian form on $H^0(\omega_{C_t})$ is positive definite.

Now suppose $g = 2$. Then N has dimension ≤ 2 and $\mathbb{P}(N)$ has dimension ≤ 1 . For each $W \subset H^0(\omega_{C_t})$ of dimension 1, the space W^\perp also has dimension 1 and hence $W \otimes W^\perp$ has dimension 1. The lines $W \otimes W^\perp$ form a real analytic subset of $\mathbb{P}(H^0(\omega_{C_t})^{\otimes 2})$ of real dimension 2. Since $\overline{\rho} : \mathcal{S} \rightarrow \overline{\mathcal{S}}$ is finite, we deduce that the lines $\rho(W \otimes W^\perp) = \mu(W \otimes W^\perp)$ form a real analytic subset of $\mathbb{P}(S^2 H^0(\omega_{C_t})) = \mathbb{P}(H^0(\omega_{C_t}^{\otimes 2})) \cong \mathbb{P}^2$ of real dimension 2. An easy computation (with coordinates) will show that this subset is not contained in any projective line in $\mathbb{P}(H^0(\omega_{C_t}^{\otimes 2}))$ and hence is not contained in $\mathbb{P}(N)$. Hence there exists W such that the line $\mu(W \otimes W^\perp)$ is not contained in N , in other words $\mu(W \otimes W^\perp) \cap N = \{0\}$. \square

We now consider the case $V \subset \mathcal{A}'_g$. As before, we first prove

Proposition 1.5. *Suppose that $g \geq 3$. Let V be a subvariety of codimension at most g of \mathcal{A}'_g . Let t be a point of V_{sm} and let N be the kernel of $\pi_a : S^2 H^0(\Omega^1_{A_t}) \rightarrow T_t^* V$. Suppose that, for any one-dimensional subvector space W of $H^0(\Omega^1_{A_t})$, the map $\pi_a \rho : W \otimes H^0(\Omega^1_{A_t}) \rightarrow T_t^* V$ is not injective. Then V has codimension exactly g and there is a one-dimensional subvector space W_N of $H^0(\Omega^1_{A_t})$ such that $N = \rho(W_N \otimes H^0(\Omega^1_{A_t}))$.*

Proof: If the map $\pi_a \rho : W \otimes H^0(\Omega^1_{A_t}) \rightarrow T_t^* V$ is not injective, then $\rho(W \otimes H^0(\Omega^1_{A_t})) \cap N \neq \{0\}$. If this holds for every $W \subset H^0(\Omega^1_{A_t})$ of dimension 1, then $\mathbb{P}(N)$ has dimension $g-1$ and is contained in

$\overline{\mathcal{S}} \cong S^2\mathbb{P}(H^0(\Omega_A^1))$. It follows that V has codimension g . The rest of the argument is now analogous to the proof of part 1 of Proposition 1.1 with $N' = N$. \square

Proof of Theorem 1 in the case of abelian varieties: This proof is now as in the case of curves. \square

Proof of Corollary 1:

Let V be a complete subvariety of codimension $g - d$ ($d \geq 0$) of \mathcal{A}_g . By Theorem 1, the set $E_1(V)$ is dense in V . In particular, it is nonempty. Let Y be an irreducible component of $E_1(V)$. Let r and s be integers such that for every ppav A with moduli point in Y there is an elliptic curve E , a ppav B and an isogeny $\nu : E \times B \rightarrow A$ of degree at most r such that the inverse image of the principal polarization of A by ν is a polarization of degree at most s . Let Y' be an irreducible component of the variety parametrizing such quadruples (E, B, A, ν) . Then Y' is a finite cover of Y . The morphism $Y' \rightarrow \mathcal{A}_1$ which to (E, B, A, ν) associates the isomorphism class of E is constant since Y' is complete (and irreducible) and \mathcal{A}_1 is affine.

For any irreducible component Z of $E_1(\mathcal{A}_g)$, there is a finite correspondance between Z and $\mathcal{A}_{g-1} \times \mathcal{A}_1$. In particular, the codimension of Z in \mathcal{A}_g is $\frac{g(g+1)}{2} - (\frac{g(g-1)}{2} + 1) = g - 1$. The variety Y is an irreducible component of the intersection of V with such a Z , hence there is a nonnegative integer e_0 such that the codimension of Y in V is $g - 1 - e_0$. So the codimension of Y in \mathcal{A}_g is $g - d + g - 1 - e_0 = 2g - d - 1 - e_0$. Since Y' maps to a point in \mathcal{A}_1 , its image V_1 in \mathcal{A}_{g-1} by the second projection has dimension equal to the dimension of Y . Therefore V_1 has dimension $g(g+1)/2 - (2g - d - 1 - e_0) = (g-1)g/2 - (g-1-d-e_0)$, i.e., codimension $g-1-d-e_0 \leq g-1$ in \mathcal{A}_{g-1} . By Theorem 1, the set $E_1(V_1)$ is dense in V_1 . In particular, the set $E_1(V_1)$ is nonempty. Let Y_1 be an irreducible component of $E_1(V_1)$ and let Y'_1 be the analogue of Y' for Y_1 . Then, as before, the variety Y_1 has codimension $g-1-d-e_0+g-2-e_1$ in \mathcal{A}_{g-1} (for some nonnegative integer e_1), the variety Y'_1 maps to a point in \mathcal{A}_1 and its image V_2 in \mathcal{A}_{g-2} has codimension $g-2-d-e_0-e_1$. Repeating the argument, we obtain V_i in \mathcal{A}_{g-i} of codimension $g-i-d-e_0-\dots-e_{i-1}$ containing Y_i of codimension $g-i-d-e_0-\dots-e_{i-1}+g-i-1-e_i$ in \mathcal{A}_{g-i} . For $i = g-2$, we can repeat the argument one last time for $V_{g-2} \subset \mathcal{A}_2$ to obtain Y'_{g-2} with image V_{g-1} in \mathcal{A}_1 with codimension $1-d-e_0-\dots-e_{g-2}$. Since \mathcal{A}_1 is affine, the variety V_{g-1} is a point and $d = e_0 = \dots = e_{g-2} = 0$. Therefore Y has codimension $2g-1$ in \mathcal{A}_g , all the varieties Y_i

have codimension $g - i + g - i - 1 = 2g - 2i - 1$ in \mathcal{A}_{g-i} , V has codimension g in \mathcal{A}_g and V_i has codimension $g - i$ in \mathcal{A}_{g-i} . In particular, the first part of Corollary 1 is proved.

For each i , there is an irreducible subvariety Z_i of V which parametrizes ppav's isogenous to the product of an element of V_i and i fixed elliptic curves ($Z_1 = Y$) because all the maps $Y'_i \longrightarrow \mathcal{A}_1$ (and also $Y' \longrightarrow \mathcal{A}_1$) are constant. It follows from the above that Z_i has the expected dimension $\frac{(g-i)(g-i+1)}{2} + i - g$. Since our choices of the Y_i 's (and Y) and hence our choices of the Z_i 's were arbitrary, we have proved the second part of the Corollary as well.

To prove the third part, first observe that a dimension count (similar to the case of Y) shows that the dimension of any irreducible component X of $E_q(V)$ is at least $\frac{q(q+1)}{2} + \frac{(g-q)(g-q+1)}{2} - g$. Let X' be the analogue of Y' for X . Then the images X_q and X_{g-q} of X' by the two projections to \mathcal{A}_q and \mathcal{A}_{g-q} are complete subvarieties of \mathcal{A}_q and \mathcal{A}_{g-q} whose codimensions are at least q and $g - q$ respectively by part 1 of the Corollary. So we have

$$\begin{aligned} \frac{q(q+1)}{2} + \frac{(g-q)(g-q+1)}{2} - g &\leq \dim(X) = \dim(X') = \dim(X_q) + \dim(X_{g-q}) \leq \\ &\leq \frac{q(q+1)}{2} - q + \frac{(g-q)(g-q+1)}{2} - (g - q) = \frac{q(q+1)}{2} + \frac{(g-q)(g-q+1)}{2} - g. \end{aligned}$$

Therefore we have equality everywhere and part 3 is proved.

Now let V' be the analytic closure of $E_{1,g}(V)$ in V . Since, by Theorem 1, the set $E_1(V_{g-2})$ is dense in V_{g-2} (which is a curve), we see that V' contains Z_{g-2} . Since all of our choices for the Y_i and Y (and hence for the Z_i) were arbitrary, we see that V' contains $E_{1,g-2}(V)$. Repeating this reasoning, we see that V' contains $E_{1,i}(V)$ for all i , hence V' contains $E_1(V)$ and $V' = V$ by Theorem 1. \square

Proof of Corollary 2:

Let V be a complete codimension g subvariety of $\widetilde{\mathcal{M}}'_g$ or \mathcal{A}'_g . Again, by Theorem 1, the set $E_1(V)$ is nonempty. Let $Y \subset V$ be an irreducible component of $E_1(V)$ and define Y' as in the proof of Corollary 1. As in loc. cit. the variety Y is a complete subvariety of V , of codimension at most $g - 1$ in V (codimension exactly $g - 1$ by Corollary 1 if $V \subset \mathcal{A}'_g$).

Suppose that $V \subset \widetilde{\mathcal{M}}'_g$. Again, since Y' is irreducible and complete and \mathcal{A}_1 affine, the map $Y' \longrightarrow \mathcal{A}_1$ is constant, hence its differential has rank 0 everywhere. It follows from [3] pages 172-173 that, for all $t \in Y \cap V_0$ and every one-dimensional subvector space W of $H^0(\omega_{C_t})$, the map $\mu : W \otimes H^0(\omega_{C_t}) \longrightarrow T_t^*V$ is *not* injective. Since this noninjectivity is a closed condition and $E_1(V_0)$ is dense in V_0 , it follows that it holds for all $t \in V_0$.

Therefore, by Proposition 1.1 and with the notation there, for all $t \in V_0$, there is a one-dimensional subvector space W_N of $H^0(\omega_{C_t})$ such that $N = \mu(W_N \otimes H^0(\omega_{C_t}))$.

Let us globalize the constructions in the proof of Proposition 1.1. Let F_0 be the Hodge bundle on V_0 and let $S^2\mathbb{P}(F_0)$ be the quotient of the fiber product $\mathbb{P}(F_0) \times_{V_0} \mathbb{P}(F_0)$ by the involution σ exchanging the two factors of the fiber product. Let $T^*\mathcal{M}'_g$ be the cotangent bundle of \mathcal{M}'_g and let $\mathcal{N}_0 \subset T^*\mathcal{M}'_g|_{V_0}$ be the conormal bundle to V_0 . Denote by \mathcal{N}'' (resp. \mathcal{N}') the subcone of decomposable tensors (resp. rank 2 symmetric tensors) in $F_0 \otimes F_0$ (resp. S^2F_0) lying in the inverse image of \mathcal{N}_0 by the multiplication map $S^2F_0 \rightarrow T^*\mathcal{M}'_g|_{V_0}$. Then, by Proposition 1.1 and with the notation there, the fibers of \mathcal{N}'' , \mathcal{N}' , and \mathcal{N}_0 at t are respectively $W_N \otimes H^0(\omega_{C_t}) \cup H^0(\omega_{C_t}) \otimes W_N$, $\rho(W_N \otimes H^0(\omega_{C_t}))$ and $\mu(W_N \otimes H^0(\omega_{C_t}))$. Hence the morphism $m : \mathcal{N}' \rightarrow \mathcal{N}_0$ is an isomorphism because it is an isomorphism on each fiber and the map $\mathbb{P}(\mathcal{N}'') \rightarrow \mathbb{P}(\mathcal{N}_0)$ is a double cover which splits on each fiber. Since the double cover of V_0 parametrizing the rulings of the fibers of $\mathbb{P}(F_0) \times_{V_0} \mathbb{P}(F_0)$ over V_0 is split, the double cover $\mathbb{P}(\mathcal{N}'') \rightarrow \mathbb{P}(\mathcal{N}') \cong \mathbb{P}(\mathcal{N}_0)$ is globally split and hence the variety $\mathbb{P}(\mathcal{N}'')$ is the union of two subvarieties of $\mathbb{P}(F_0) \times_{V_0} \mathbb{P}(F_0)$ exchanged by σ and both isomorphic to $\mathbb{P}(\mathcal{N}')$ (by the quotient morphism $\mathbb{P}(F_0) \times_{V_0} \mathbb{P}(F_0) \rightarrow S^2\mathbb{P}(F_0)$) and to $\mathbb{P}(F_0)$ by either of the two projections $\mathbb{P}(F_0) \times_{V_0} \mathbb{P}(F_0) \rightarrow \mathbb{P}(F_0)$. In particular, the two components of $\mathbb{P}(\mathcal{N}'')$ are projective bundles on V_0 and \mathcal{N}'' is the union of two vector bundles \mathcal{N}''_1 and \mathcal{N}''_2 with respective fibers $W_N \otimes H^0(\omega_{C_t})$ and $H^0(\omega_{C_t}) \otimes W_N$ at t . Furthermore, we have $\mathcal{N}''_1 \xrightarrow{\cong} \mathcal{N}_0 \xleftarrow{\cong} \mathcal{N}''_2$ (checked on fibers again). Since $\mathbb{P}(\mathcal{N}''_1)$ is isomorphic to $\mathbb{P}(F_0)$, there is a line bundle \mathcal{W} such that $\mathcal{N}''_1 \cong \mathcal{W} \otimes F_0$. So $\mathcal{N}_0 \cong \mathcal{W} \otimes F_0$.

From the injection $\mathcal{N}''_1 \hookrightarrow F_0 \otimes F_0$ we deduce the injection $\mathcal{W} \hookrightarrow F_0$ which is the composition of the morphism $\mathcal{W} \hookrightarrow F_0 \otimes F_0 \otimes F_0^*$ (obtained from $\mathcal{W} \otimes F_0 \cong \mathcal{N}''_1 \hookrightarrow F_0 \otimes F_0$) with the morphism $F_0 \otimes (F_0 \otimes F_0^*) \xrightarrow{id \otimes tr} F_0$ which is the product of the identity $F_0 \xrightarrow{id} F_0$ and the trace morphism $F_0 \otimes F_0^* \cong End(F_0) \xrightarrow{tr} \mathcal{O}_{V_0}$.

For $V \subset \mathcal{A}'_g$ the proof is similar to (and simpler than) the above and uses Proposition 1.5 instead of Proposition 1.1. \square

2. APPENDIX: A REMARK ON DENSITY IN POSITIVE CHARACTERISTIC

In this section we use the notation of the introduction to denote moduli spaces of curves and abelian varieties over an algebraically closed field k of characteristic $p > 0$. The subvariety V_0 of \mathcal{A}_g parametrizing ppav's of p -rank 0 is a complete (connected if $g > 1$ by [11] (2.6)(c)) subvariety of codimension g of \mathcal{A}_g (see [13], (2) in the introduction and [10], the proof of Theorem 1.1a pages 98-99). We explain below how to deduce from the results of [4], [6], [9] and [11] that the moduli points of non-simple abelian varieties in V_0 are contained in a proper closed subset of V_0 when $g \geq 3$.

The formal group of an abelian variety A of p -rank 0 is isogenous to a sum

$$\sum_{1 \leq i \leq r} G_{m_i, n_i}$$

where m_i and n_i are relatively prime positive integers for each i , the sum $m_i + n_i$ is less than or equal to g for all i , the formal group G_{m_i, n_i} has dimension m_i and its dual is G_{n_i, m_i} (see [9] chapter IV, §2). The decomposition is symmetric, i.e., the group G_{m_i, n_i} appears as many times as G_{n_i, m_i} . We call the unordered r -tuple $((m_i, n_i))_{1 \leq i \leq r}$ the formal isogeny type of the abelian variety. As in [11], we define the Symmetric Newton Polygon of A to be the lower convex polygon in the plane \mathbb{R}^2 which starts at $(0, 0)$ and ends at $(2g, g)$, whose break-points have integer coordinates and whose slopes (arranged in increasing order because of lower convexity) are $\lambda_i = \frac{m_i}{m_i + n_i}$ with multiplicity $m_i + n_i$ (i.e., on the polygon, the x -coordinate grows by $m_i + n_i$ and the y -coordinate grows by m_i). The polygon is symmetric in the sense that if the slope λ appears, then the slope $1 - \lambda$ appears with the same multiplicity. Following [11], we shall say that the Newton Polygon β is above the Newton Polygon α if for all real numbers $x \in [0, 2g], y, z \in [0, g]$ such that $(x, z) \in \beta, (x, y) \in \alpha$, we have $z \geq y$. We shall say that β is strictly above α if β is above α and $\beta \neq \alpha$. Again as in [11], for a Symmetric Newton Polygon α , we denote by W_α the set of points in \mathcal{A}_g corresponding to abelian varieties whose Newton Polygon is above α . By [4] page 91, Newton polygons go up under specialization. By [6] page 143 Theorem 2.3.1 and Corollary 2.3.2 (see also [11], 2.4), for any Newton polygon α , the set W_α is closed in V_0 . By [11] Theorem (2.6)(a) and Remark (3.3), the abelian variety A_0 with moduli point the generic point of V_0 has formal isogeny type $((1, g - 1), (g - 1, 1))$. Therefore, since $g \geq 3$, the abelian variety A_0 is simple. Let α_0 denote the Symmetric Newton Polygon of A_0 . The moduli point of a non-simple ppav of p -rank 0 is in W_β for some Symmetric Newton Polygon β strictly above α_0 . Therefore the set of non-simple ppav's in V_0 is contained in $\cup_{\beta \text{ strictly above } \alpha_0} W_\beta$. Since there are only a finite number of Symmetric Newton Polygons (below the line $x = 2y$ and) above α_0 , we deduce that all points of V_0 corresponding to nonsimple abelian varieties are in a proper closed subset of V_0 (which is $\cup_{\beta \text{ strictly above } \alpha_0} W_\beta$).

Therefore V_0 is an example of a subvariety V of codimension g of \mathcal{A}_g (for all $g \geq 3$) or of $\widetilde{\mathcal{M}}_3$ such that $E_q(V)$ is not Zariski-dense in V for any q .

REFERENCES

1. A. Andreotti and A. Mayer, *On period relations for abelian integrals on algebraic curves*, Ann. Scuola Norm. Sup. Pisa **21** (1967), 189–238.

2. E. Arbarello, M. Cornalba, P.A. Griffiths, and J. Harris, *Geometry of algebraic curves*, vol. 1, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1985.
3. E. Colombo and G.P. Pirola, *Some density results for curves with non simple jacobians*, Math. Annalen **288** (1990), 161–178.
4. M. Demazure, *Lectures on p -divisible groups*, Lecture Notes in Math., vol. 302, Springer-Verlag, 1972.
5. S. Diaz, *Complete subvarieties of the moduli space of smooth curves*, Algebraic Geometry, Bowdoin 1985, Proceedings of Symposia in Pure Mathematics, vol. 46, American Mathematical Society, 1987, pp. 77–81.
6. N.M. Katz, *Slope filtrations of F -crystals*, Astérisque **63** (1979), 113–164.
7. G. van der Geer, *Cycles on the moduli space of abelian varieties*, Algebraic Geometry Preprint, alg-geom/9605011.
8. E. Looijenga, *Smooth Deligne-Mumford compactifications by means of Prym-level structures*, Journal of Algebraic Geometry **3** (1994), 283–293.
9. Y. I. Manin, *The theory of commutative formal groups over fields of finite characteristic*, Russian Math. Surveys **18** (1963), 1–83.
10. F. Oort, *Subvarieties of moduli spaces*, Inventiones Math. **24** (1974), 95–119.
11. ———, *Moduli of abelian varieties and Newton polygons*, C. R. Acad. Sci. Paris **312** (1991), 385–389.
12. ———, *Complete subvarieties of moduli spaces*, Abelian Varieties, Proceedings of the 1993 International Conference at Egloffstein, de Gruyter, 1995, pp. 225–235.
13. F. Oort and P. Norman, *Moduli of abelian varieties*, Annals of Math. **112** (1980), 413–439.
14. M. Pikaart and A.J. de Jong, *Moduli of curves with non-abelian level structure*, The Moduli Spaces of Curves, Proceedings of the 1994 Texel conference (R. Dijkgraaf, C. Faber, and G. van der Geer, eds.), Progress in Mathematics, vol. 129, Birkhäuser, 1995, pp. 483–509.
15. J. Tate, *Classes d'isogénie des variétés abéliennes sur un corps fini (d'après T. Honda)*, Séminaire Bourbaki **21** (1968/69), Exposé 352.

DEPARTMENT OF MATHEMATICS, BOYD GRADUATE STUDIES RESEARCH CENTER, UNIVERSITY OF GEORGIA,
ATHENS, GA 30602-7403

E-mail address: izadi@math.uga.edu