

# The moduli space of étale double covers of genus 5 curves is unirational

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## Abstract

We show that the coarse moduli space  $\mathcal{R}_5$  of étale double covers of curves of genus 5 over the complex numbers is unirational. We give two slightly different arguments, one purely geometric and the other more computational.

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## 0 Introduction

The coarse moduli space  $\mathcal{R}_g$  of étale double covers of genus  $g$  curves is sometimes referred to as the Prym moduli space. It can be thought of as the moduli space of curves  $C \in \mathcal{M}_g$  equipped with a nontrivial line bundle  $\mathcal{L}$  whose square is trivial. Thus  $\mathcal{R}_g$  is also equipped with a morphism to  $\mathcal{M}_g$ , which is a finite cover of degree  $2^{2g} - 1$ . It has been extensively studied for small values of  $g$ . In particular Donagi showed in [10] that  $\mathcal{R}_6$  is unirational. Other proofs of the unirationality of  $\mathcal{R}_6$  were given by Verra [17] and by Mori and Mukai [16].

For  $g \leq 6$  the Prym map  $p_g: \mathcal{R}_g \rightarrow \mathcal{A}_{g-1}$ , which associates to an étale double cover  $\tau: \tilde{C} \rightarrow C$  the Prym variety  $P(\tau) = \text{coker}(\tau^*: \text{Jac } C \rightarrow \text{Jac } \tilde{C})$ , is dominant. It therefore follows from Donagi's result that the moduli space  $\mathcal{A}_5$  of principally polarised abelian 5-folds is unirational.

Catanese [6] showed that  $\mathcal{R}_4$  is rational. The rationality of  $\mathcal{R}_3$  could be attributed to Katsylo [12], Bardelli, Del Centina, Recillas [2] and [8], and Dolgachev [9]. The rationality of  $\mathcal{R}_2$  is also due to Dolgachev [9]. Moreover  $\mathcal{R}_1 = X_0(2)$  is rational.

Clemens [7] showed that  $\mathcal{A}_4$  is unirational, but using intermediate Jacobians, not Prym varieties. In the introduction to [6] it is stated that  $\mathcal{R}_5$  is unirational and a reference is given to [7], then unpublished; but as far as we can determine no proof is given there or anywhere else.

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In this paper we fill this gap by proving (Theorem 5.1) that  $\mathcal{R}_5$  is indeed unirational. We work over an algebraically closed field  $\mathbb{K}$  of characteristic different from 2 (except in Section 4).

The basic construction used in our proof is to be found in [7]. If  $X$  is a quartic surface in  $\mathbb{P}^3$  with six ordinary double points at  $P_0, \dots, P_5$  and no other singularities we define  $C_X$  to be the discriminant of the projection from  $P_0$ . Generically it is a 5-nodal plane sextic (hence of genus 5), with an everywhere tangent conic coming from the tangent cone to  $X$  at  $P_0$ . The quartic double solid branched along  $X$  has (after blowing up  $P_0$ ) the structure of a conic bundle over  $\mathbb{P}^2$  with discriminant curve  $C_X$ . Blowing up in the remaining five points yields a conic bundle over a degree 4 del Pezzo surface, and the discriminant is the canonical model  $\tilde{C}_X$  of  $C_X$ . This determines a connected étale double cover of  $\tilde{C}_X$ .

The space  $\mathcal{Q}$  of quartic surfaces in  $\mathbb{P}^3$  with six isolated ordinary double points, one of which is marked, is unirational. This is well-known and quite easy to prove: see Proposition 2.1.

The construction above defines a morphism from the unirational variety  $\mathcal{Q}$  to  $\mathcal{R}_5$ , which is in turn endowed with a finite (in fact 1024-to-1) natural projection to  $\mathcal{M}_5$ . Since  $\mathcal{R}_5$  is irreducible, to prove the unirationality it is now enough to prove that the map to  $\mathcal{M}_5$  is generically surjective.

We present two different proofs that the map  $\theta: \mathcal{Q} \rightarrow \mathcal{M}_5$  is dominant. One method exploits the special geometry of the family, using ideas of Donagi as worked out in [11]. We show by a dimension count that the general genus 5 curve does have a plane model as a 5-nodal sextic with an everywhere tangent conic, and then show how to recover the quartic in  $\mathbb{P}^3$  as a certain image of the double cover of  $\mathbb{P}^2$  branched along the sextic.

The other approach, which was used in [15], is computational, and is applicable to any family of 5-nodal sextics. It uses the fact that  $\tilde{C}_X$  is a canonical curve, and reduces the question of surjectivity of the Kodaira-Spencer map to computing the rank of a certain matrix. This can then be verified at a test point.

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# 1 Nodal quartics and nodal curves

The equation of a quartic surface  $X \subset \mathbb{P}^3$  with an isolated ordinary double point at  $P_0 = (0 : 0 : 0 : 1)$  is  $F = u_2x_3^2 + 2u_3x_3 + u_4 = 0$ , where  $u_d$  is a form of degree  $d$  in  $\mathbb{K}[x_0, x_1, x_2]$  and the quadratic form  $u_2$  is non-degenerate. The projection  $\pi: X \setminus P_0 \rightarrow \mathbb{P}^2$  from  $P_0$  is induced by the homomorphism

$$r: \mathbb{K}[x_0, x_1, x_2] \rightarrow \mathbb{K}[x_0, \dots, x_3]/(F),$$

sending  $x_i$  to  $x_i + (F)$  for  $i = 0, 1, 2$ . If  $X$  is general, then any line in  $\mathbb{P}^3$  through  $P_0$  intersects  $X$  in at most two other points so  $\pi$  is a quasi-finite morphism.

We define the plane curve  $C_X$  to be the locus of lines through  $P_0$  tangent to  $X$  away from  $P_0$ . The following is easy to prove: see [13, Lemma 5.1].

**Proposition 1.1**  *$C_X$  is a plane curve of degree six given by  $u_3^2 - u_2u_4 = 0$ . Furthermore if  $Q \in X$  is a singular point (different from  $P$ ), then  $\pi(Q)$  is a singular point of  $C_X$ .*

The locus of points in  $\mathbb{P}^2$  whose reduced fibre under  $\pi$  consists of only one point is not irreducible. There are two components, the curve  $C_X$  and the conic  $u_2 = 0$ .

**Definition 1.2** *A conic  $V \subset \mathbb{P}^2$  is called a contact conic of  $C_X$  if  $V$  cuts on  $X$  a divisor which is divisible by 2.*

**Corollary 1.3** *Let  $\text{Bl}_{P_0}(X) \rightarrow X$  be the blow-up of the surface  $X$  at the point  $P_0$ . Then the unique morphism  $\text{Bl}_{P_0}(X) \rightarrow \mathbb{P}^2$  which extends  $\pi$  is a double cover of  $\mathbb{P}^2$  branched along  $C_X$ . The image of the exceptional divisor in  $\text{Bl}_{P_0}(X)$  is the contact conic of  $C_X$  defined by the equation  $u_2 = 0$ .*

*Proof.* It is essentially enough to observe that the tangent cone of  $X$  at  $P_0$  is defined by the equation  $u_2 = 0$  in  $\mathbb{P}^3$ . The exceptional curve inside  $\text{Bl}_{P_0}(X)$  corresponds to the set of lines in the tangent cone. To see that the conic  $u_2 = 0$  is a contact conic of  $C_X$  simply look at the ideal of the intersection of the conic with  $C_X$ :  $(u_2, u_3^2 - u_2u_4) = (u_2, u_3^2)$  in  $\mathbb{K}[x_0, x_1, x_2]$ . In particular this means that the points of contact are given by  $u_2 = u_3 = 0$   $\square$

Next we describe the singular locus of  $C_X$ .

**Lemma 1.4** *Let  $Y_d \subset \mathbb{P}^3$  be the cone of vertex  $P_0$  defined by the form  $u_d$ , and let  $Q \in X$  be any point different from  $P_0$  such that  $\pi(Q) \in C_X$ . Then  $\pi(Q) \in \text{Sing } C_X$  if and only if  $Q \in (\text{Sing } X) \cup (Y_2 \cap Y_3 \cap Y_4)$ .*

*Proof.* If  $\mathfrak{q}$  is the homogeneous ideal of  $Q$ , then  $Q \in X$  means  $F \in \mathfrak{q}$  and  $\pi(Q) \in C_X$  means  $u_3^2 - u_2u_4 \in \mathfrak{q}$ . Then, from the equality

$$u_2F = (u_2x_3 + u_3)^2 - (u_3^2 - u_2u_4)$$

we obtain immediately  $u_2x_3 + u_3 \in \mathfrak{q}$ . Now  $\pi(Q)$  is a singular point if and only if  $u_3^2 - u_2u_4 \in \mathfrak{q}^2$ , and this happens if and only if  $u_2F \in \mathfrak{q}^2$ , which means that either  $u_2 \in \mathfrak{q}$  or  $F \in \mathfrak{q}^2$ . In the latter case  $Q$  is a singular point of  $X$ ; in the former we also have  $u_3 \in \mathfrak{q}$ , since  $u_2x_3 + u_3 \in \mathfrak{q}$ , and  $u_4 \in \mathfrak{q}$  since  $2u_3x_3 + u_4 \in \mathfrak{q}$ .  $\square$

**Proposition 1.5** *For a general quartic surface  $X$  with an isolated double point  $P_0$ , the singular points of  $C_X$  are all images of singular points of  $X$ .*

*Proof.* Suppose that  $Q$  is a point of  $X$  mapping to a singular point of  $C_X$  and  $P_0 \neq Q \in Y_2 \cap Y_3 \cap Y_4$ . Then  $\pi(Q) \in (u_2 = u_3 = u_4 = 0)$ , which is empty for general  $(u_2, u_3, u_4)$ .  $\square$

If  $X$  is a quartic surface with at least one ordinary double point  $P_0$ , we let  $p: \Lambda_X \rightarrow \mathbb{P}^3$  be the double cover branched along  $X$  and let  $W_X = \text{Bl}_{P_0}(\Lambda_X)$  be the blow-up of  $\Lambda_X$  at  $P_0$ .

**Proposition 1.6** *Let  $X, P_0, \Lambda_X$  and  $W_X$  be as above. The unique morphism  $f$  that makes the diagram*

$$\begin{array}{ccc} W_X & \longrightarrow & \Lambda_X \xrightarrow{p} \mathbb{P}^3 \\ & \searrow f & \downarrow \pi \\ & & \mathbb{P}^2 \end{array}$$

*commute is a conic bundle over  $\mathbb{P}^2$ , and the curve  $C_X$  is the locus of points whose fibre is a degenerate conic.*

*Proof.* This is shown in [7, p.222]: there is a more detailed version in [14, Section 2].  $\square$

Kreussler [14, Section 2] gives an explicit equation for  $W_X$  as a divisor inside a  $\mathbb{P}^2$  bundle over  $\mathbb{P}^2$ . Put  $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-2)$  over  $\mathbb{P}^2$ , and consider  $p: \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^2$ . Here  $\mathbb{P}(E)$  is the projective bundle of hyperplanes in the fibres of  $E$ . Let  $z_k \in H^0(\mathbb{P}^2, \mathcal{E}(k))$ ,  $k = 0, 1, 2$  be *constant* non-zero global sections and define the divisor  $W_X \subset \mathbb{P}(\mathcal{E})$  by

$$-z_2^2 + z_1^2u_2 + 2z_1z_0u_3 + z_0^2u_4 = 0$$

(the left-hand side is a section of  $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes p^*\mathcal{O}_{\mathbb{P}^2}(4)$ ).

**Lemma 1.7** *Let  $C$  be a plane sextic curve whose only singularities are five nodes in linear general position. Let  $\sigma_C: \widetilde{\mathbb{P}^2} \rightarrow \mathbb{P}^2$  denote the blow-up of  $\mathbb{P}^2$  in these five points. Then  $\widetilde{\mathbb{P}^2}$  is a degree 4 Del Pezzo surface and the anticanonical embedding of  $\widetilde{\mathbb{P}^2}$  in  $\mathbb{P}^4$  realises the strict transform  $\widetilde{C}$  of  $C$  as a smooth canonically embedded curve of genus 5.*

*Proof.* This is well known. That  $\widetilde{C}$  is canonically embedded follows from a simple adjunction computation: if  $E$  is the exceptional divisor and  $H$  is the class of a line in  $\mathbb{P}^2$ , then in  $\text{Pic } \widetilde{\mathbb{P}^2}$  we have  $\widetilde{C} = \sigma_C^*(6H) - 2E = -2K_{\widetilde{\mathbb{P}^2}}$ .  $\square$

Next we consider a more special case. We suppose  $X \subset \mathbb{P}^3$  is a quartic surface with six isolated ordinary double points,  $P_0, \dots, P_5$  and no other singularities. We also assume that the  $P_i$  are in linear general position and that  $X$  is general with respect to  $P_0$ , in the precise sense of Proposition 1.5. Under these hypotheses  $C_X$  is a plane sextic with precisely five nodes, at  $\overline{P}_i = \pi(P_i)$ ,  $1 \leq i \leq 5$ .

Let  $f: W_X \rightarrow \mathbb{P}^2$  be the conic bundle as in Proposition 1.6, and let  $\sigma_{C_X}: \widetilde{\mathbb{P}^2} \rightarrow \mathbb{P}^2$  and  $\widetilde{C}_X$  be as in Lemma 1.7. Let  $\Sigma$  be the surface  $f^{-1}(C_X) \subset W_X$  and put  $S = \Sigma \times_{C_X} \widetilde{C}_X$ , with  $\tilde{f}: S \rightarrow \widetilde{C}_X$  the projection.

Let  $\nu: \widetilde{S} \rightarrow S$  be the normalisation of  $S$ , and consider the Stein factorisation of  $\tilde{f} \circ \nu: \widetilde{S} \rightarrow \widetilde{C}_X$

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\nu} & S \\ f' \downarrow & & \downarrow \tilde{f} \\ \Gamma & \xrightarrow{g} & \widetilde{C}_X \end{array}$$

so  $f'$  is a projective morphism with connected fibres and  $g$  is a finite morphism.

**Proposition 1.8** *For general  $X$ , the finite morphism  $g: \Gamma \rightarrow \widetilde{C}_X$  is an étale degree 2 map between smooth connected curves, naturally associated with the pair  $(X, P_0)$ .*

*Proof.* The cover  $g$  is unbranched of degree 2 because the restriction of the conic bundle  $W_X$  to the curve  $C_X$  consists of a fibration by pairs of distinct lines. To see this recall the equation for the conic bundle  $W_X$ . The preimage of  $x = (x_1 : x_2 : x_3) \in \mathbb{P}^2$  is given by

$$-z_2^2 + z_1^2 u_2(x) + 2z_1 z_0 u_3(x) + z_0^2 u_4(x) = 0,$$

and this has rank 2 since  $u_2, u_3$  and  $u_4$  never vanish simultaneously.

It remains to check that  $g$  is nontrivial, that is, that  $\Gamma$  is connected. In characteristic zero this follows from the fact that the Prym variety  $P(g)$  is isomorphic to the intermediate Jacobian of the conic bundle [5, Section 2]. If the double cover were trivial, then  $P(g)$  would have dimension 5, which

is impossible. To extend this to the case of characteristic  $p \neq 2$ , we observe that the quartic equation lifts to characteristic zero and in that case the double cover is nontrivial, as we have just seen. Therefore the double cover in positive characteristic is connected.  $\square$

## 2 Moduli of curves

The functor  $\mathbf{r}_g$  ( $g \geq 2$ ) given by families of smooth projective curves of genus  $g$  with a connected étale double cover is coarsely represented by an irreducible quasi-projective scheme  $\mathcal{R}_g$ : see [4, §6]. The dimension of this moduli space is  $3g - 3$ , the same as the dimension of  $\mathcal{M}_g$ , the extra data being one of the  $2^{2g} - 1$  nontrivial 2-torsion points in the Jacobian of  $C$ . So forgetting this defines a natural transformation between  $\mathbf{r}_g$  and  $\mathbf{m}_g$ , and thus a morphism  $\mathcal{R}_g \rightarrow \mathcal{M}_g$  with finite fibres.

Fix five points  $P_1, \dots, P_5$  in  $\mathbb{P}^3$  in linear general position. Let  $\mathcal{Q}$  be the space of quartic surfaces in  $\mathbb{P}^3$  with ordinary double points at the  $P_i$ , one additional ordinary double point distinct from the  $P_i$ , and no other singularity.

**Proposition 2.1**  *$\mathcal{Q}$  is an irreducible locally closed subscheme of the Hilbert scheme of quartic surfaces in  $\mathbb{P}^3$  hence inherits a universal family of quartics from the Hilbert scheme. Furthermore  $\mathcal{Q}$  is unirational of dimension 13.*

*Proof.* The Hilbert scheme of quartic surfaces in  $\mathbb{P}^3$  is  $\text{Hilb}_{2m^2+2}(\mathbb{P}^3) = \mathbb{P}(H^0(\mathbb{P}^3, \mathcal{O}(4)))$ .

Let  $\mathfrak{p}_i$  be the homogeneous ideal of  $P_i$  in  $\mathbb{P}^3$ . The set of quartic surfaces in  $\mathbb{P}^3$  with five double points at the  $P_i$  is the closed subscheme of the Hilbert scheme given by  $\mathbb{P}(I_4)$  where  $I = \bigcap_{i=0}^4 \mathfrak{p}_i^2$  is the ideal of the five double points. In other words it is  $\mathbb{P}H^0(\mathbb{P}^3, \mathcal{I}(4))$ , where we denote  $\mathcal{I}$  the sheaf of ideals defined by  $I$ . Note that  $h^0(\mathbb{P}^3, \mathcal{I}(4)) = 15$ .

Going on, we ask now for a sixth double point. We take the product  $\mathbb{P}(I_4) \times \mathbb{P}^3$  and consider the closed subscheme  $B_0$  defined as

$$B_0 := \{(F, \mathfrak{p}_0) \mid F \in \mathfrak{p}_0^2\} = \left\{ (F, P_0) \mid F(P_0) = \frac{\partial F}{\partial x_i}(P_0) = 0 \quad \forall i \right\}.$$

The projection

$$B_0 \longrightarrow \mathbb{P}^3$$

is surjective and the fibres are linear spaces. Indeed, having a singular point at  $P_0 \in \mathbb{P}^3$  defines four linear conditions on the linear space  $H^0(\mathbb{P}^3, \mathcal{I}(4))$ . Over an open subset  $U \subset \mathbb{P}^3$  these conditions are independent and define a vector bundle  $E$  of rank 11. To see that  $U$  is not empty it is enough to fix a sixth point in  $\mathbb{P}^3$  and compute the Hilbert function of the ideal  $J = \bigcap_{i=0}^5 \mathfrak{p}_i^2$ , which we may easily do with *Macaulay*. The projective space bundle  $B_1$

over  $U$  associated to  $E$  is a rational variety embedded in  $B_0$  as a dense open subset. Hence  $B_0$  is irreducible and rational.

Projecting onto  $\mathbb{P}(I_4)$ , we consider the scheme theoretic image of  $B_0$ , which is a closed subscheme  $B$  of  $\mathbb{P}(I_4)$ , and observe that in the universal factorisation

$$\begin{array}{ccc} B_0 & \xrightarrow{\beta} & B \\ & \searrow & \downarrow \\ & & \mathbb{P}(I_4) \end{array}$$

the dominant morphism  $\beta$  is also proper, because it is the external morphism of a composition which is proper. As a consequence  $\beta$  is surjective and the scheme theoretic image of  $B_0$  coincides with the set theoretic image. Observe also that  $B$  is irreducible. However  $B$  contains all the possible degenerations of a quartic surface with six double points, while we are interested in those surfaces with ordinary double points and no other singularities. This is clearly an open condition, so we have proved that  $\mathcal{Q}$  is an open subset of an irreducible closed subset of the Hilbert scheme.

The unirationality of  $\mathcal{Q}$  follows from the fact that an open dense subset of the rational variety  $B_0$  maps onto it.

Finally, one may check by computing the differential of the projection at one point that the dimension of  $\mathcal{Q}$  is 13 (it is irreducible because it is an open subset of an irreducible variety).  $\square$

**Proposition 2.2** *There exists a morphism of schemes  $\varrho: \mathcal{Q} \rightarrow \mathcal{R}_5$  given by the constructions above that associates to any nodal quartic surface in  $\mathbb{P}^3$  a nodal sextic plane curve with a double cover.*

*Proof.* We must globalise our earlier constructions. This is a standard gluing argument. Suppose first that the base scheme is  $B = \text{Spec } A$  for some ring  $A$ . Associate to the scheme  $\mathcal{X} = \text{Proj } A[x_0, \dots, x_3]/(u_2x_3^2 + 2u_3x_3 + u_4)$  the plane curve over  $A$  defined by the equation  $u_3^2 - u_2u_4$ . This association is natural, in that it commutes with pull-backs. Indeed for any homomorphism of rings  $A \rightarrow A'$  the pull-back of  $\mathcal{X}$  is given by  $\text{Proj}(A \otimes A')[x_0, x_1, x_2]/(u_3^2 - u_2u_4)$ , and this is the same graded ring one would obtain by first pulling back the family of surfaces and then applying the correspondence.  $\square$

We want to use the morphism  $\varrho$  to prove the unirationality of  $\mathcal{R}_5$ . To do so we must show that  $\varrho$  is dominant. We can simplify the problem by taking advantage of the irreducibility of  $\mathcal{R}_g$  and  $\mathcal{M}_g$ . We have a commutative diagram

$$\begin{array}{ccc} \mathcal{Q} & \xrightarrow{\varrho} & \mathcal{R}_5 \\ & \searrow \theta & \downarrow \eta \\ & & \mathcal{M}_5 \end{array}$$

where  $\eta$  forgets the double cover.

**Lemma 2.3** *The morphism  $\varrho$  is dominant if and only if  $\theta$  is dominant.*

*Proof.* Since  $\eta$  is a dominant morphism between irreducible spaces, it is immediate that  $\theta$  is dominant if  $\varrho$  is. Conversely, if  $\varrho$  is not dominant then the scheme theoretic image  $\varrho(\mathcal{Q}) \subset \mathcal{R}_5$  has dimension less than  $\dim \mathcal{R}_5 = 12$ , because  $\mathcal{R}_5$  is irreducible, so  $\dim \theta(\mathcal{Q}) < 12$  also, so  $\theta$  is not dominant.  $\square$

### 3 Reconstructing the double solid

In this section we give a proof that  $\theta: \mathcal{Q} \rightarrow \mathcal{M}_5$  is dominant by making use of the special geometry of the family  $\mathcal{Q}$ . We show how to reconstruct the quartic surface from a suitable plane sextic model of a sufficiently general genus 5 curve.

**Lemma 3.1** *For a general  $C \in \mathcal{M}_5$  there exists a birational map  $C \rightarrow \bar{C} \subset \mathbb{P}^2$  to a plane 5-nodal sextic  $\bar{C}$ , such that  $\bar{C}$  admits a contact conic  $V \subset \mathbb{P}^2$  meeting  $\bar{C}$  at six distinct smooth points of  $\bar{C}$ . Furthermore, a general  $C \in \mathcal{M}_5$  has a one-parameter family of such birational plane models.*

*Proof.* By the Kempf-Kleiman-Laksov theorem ([1, V (1.1)]), a general  $C \in \mathcal{M}_5$  has a 2-dimensional family  $G_6^2(C)$  of  $g_6^2$ s (linear systems of degree 6 and dimension 2) and hence of birational models  $\bar{C}$  as a plane sextic. The general such sextic, for any given general  $C$ , has five nodes.

For fixed general  $C$ , the image of the map  $|\mathcal{O}_{\mathbb{P}^2}(2)| \times G_6^2(C) \cong \mathbb{P}^5 \times \mathbb{P}^2 \rightarrow \text{Hilb}_{12} \mathbb{P}^2$  given by  $(V, \bar{C}) \mapsto V \cap \bar{C} \subset V \cong \mathbb{P}^2$  intersects the codimension 6 locus consisting of subschemes with multiplicity at least 2 at each point in a variety of dimension 1. In particular this intersection is nonempty. Indeed, if we take  $\bar{C}$  to be the discriminant curve of the projection from a node of a general 6-nodal quartic surface in  $\mathbb{P}^3$ , i.e. we take  $F$  as in Section 1 with  $u_d$  general, and  $V = (u_2 = 0)$ , we obtain a pair  $(V, \bar{C})$  whose intersection is six distinct smooth points.

Thus for a general genus 5 curve  $C$ , there is a 1-dimensional family of plane 5-nodal sextic models  $\bar{C}$  of  $C$ , each having a contact conic meeting  $\bar{C}$  at six distinct smooth points of  $\bar{C}$ .  $\square$

**Proposition 3.2** *Given  $\bar{C}$  as in Lemma 3.1 with contact conic  $V$ , there exists a quartic surface  $X$  with 6 nodes such that  $\bar{C}$  arises as the discriminant locus of the projection of  $X$  from one of its nodes and  $V$  is the projection of the tangent cone of  $X$  at the same node.*

*Proof.* We follow the construction on pages 104–105 of [11]. We take the double cover  $\psi: Y \rightarrow \mathbb{P}^2$  branched along  $\bar{C}$  and map it to  $\mathbb{P}^3$  by a linear system determined by  $V$ .

To define the linear system, take the desingularisation  $\sigma: \tilde{Y} \rightarrow Y$ . The inverse image  $\sigma^{-1}(V)$  consists of two components,  $V_+$  and  $V_-$ . We consider the linear systems  $H_{\pm} = |(\psi\sigma)^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{\tilde{Y}}(V_{\pm})|$ . Either of these linear systems (and no others) maps the K3 surface  $\tilde{Y}$  onto a quartic surface  $\bar{Y}$ , unique up to projective equivalence, with six nodes: five of these nodes are the images of the exceptional curves of  $\sigma$ , corresponding to nodes of  $Y$ , and the sixth is the image of  $V_{\pm}$ . The discriminant curve for the projection from this sixth node is  $\bar{C}$  up to projective equivalence. We refer to the proof of [11, Theorem 2.1.1] for the computations of the degree and dimension of  $H_{\pm}$  needed to justify these assertions.  $\square$

**Corollary 3.3** *The map  $\varrho: \mathcal{Q} \rightarrow \mathcal{R}_5$  is dominant.*

*Proof.* Immediate from Proposition 3.2.  $\square$

## 4 Families of canonical curves

In this section we give an alternative proof that  $\theta$ , and hence  $\varrho$ , is dominant. The method is to check directly, by computation, that the Kodaira-Spencer map is locally surjective at a test point. It does not rely on the special geometry of  $\mathcal{Q}$ : it is a method of checking computationally that a given family of 5-nodal sextics is general in the sense of moduli of genus 5 curves. For simplicity we assume in this section that  $\mathbb{K}$  is of characteristic zero.

We first write down a local condition for  $\theta$  to be dominant, i.e. generically surjective.

**Lemma 4.1** *Let  $u: X \rightarrow X'$  be a morphism, with  $X'$  irreducible. Then  $u(X)$  is Zariski dense in  $X'$  if and only if there exists a smooth point  $P \in X$  such that the differential  $du_P: T_{X,P} \rightarrow T_{X',u(P)}$  is surjective.*

*Proof.* Since  $X'$  is irreducible the closure of  $u(X)$  is  $X'$  if and only if the dimension of one of its irreducible components is equal to  $\dim X'$ . Now it is enough to recall that the dimension of the irreducible component of  $\overline{u(X)}$  containing a regular point  $u(P)$  is given by the rank of the differential.  $\square$

The tangent space to any scheme  $X$  at a closed point  $P$  is the set of maps from  $D = \text{Spec } \mathbb{K}[\varepsilon]/(\varepsilon^2)$  to  $X$  centred at  $P$ . For any morphism  $u: X \rightarrow X'$  and any closed regular point  $P \in X$  the differential  $du_P: T_{X,P} \rightarrow T_{X',u(P)}$  is given by  $\varphi \mapsto u \circ \varphi$ .

Let  $C$  be a canonically embedded curve of genus five, which we assume to be given by the complete intersection of three quadrics in  $\mathbb{P}^4$  (i.e., by Petri's Theorem, non-trigonal). Two canonically embedded curves of genus  $g$  are isomorphic if and only if they are projectively equivalent.

We put  $R_2 = H^0(\mathcal{O}_{\mathbb{P}^4}(2))$ , the degree 2 part of  $\mathbb{K}[x_0, \dots, x_4]$ , which we identify with the space of  $4 \times 4$  symmetric matrices over  $\mathbb{K}$ .

The set of all canonical curves in  $\mathbb{P}^4$  is an open subset of the Grassmannian  $\text{Gr}(3, R_2)$ . Projective equivalence is then given by the action of the group  $\text{PGL}(5)$  on  $\mathbb{P}^4$ .

But the Grassmannian itself can be realised as an orbit space, this time under the action of  $\text{GL}(3)$ , as follows. Let  $V$  be the open set inside the 45-dimensional vector space  $R_2 \times R_2 \times R_2$  where the three components span a 3-dimensional subspace of  $R_2$ , and consider the action of  $\text{GL}(3)$  whose orbits are all the possible bases for a given subspace. This is the action

$$M({}^t\underline{x}Q_1\underline{x}, {}^t\underline{x}Q_2\underline{x}, {}^t\underline{x}Q_3\underline{x})_j = \sum_{i=1}^3 m_{ji} {}^t\underline{x}Q_i\underline{x}, \quad 1 \leq j \leq 3,$$

where  $M \in \text{GL}(3)$  and  ${}^t\underline{x}$  is the row vector  $(x_0, \dots, x_4)$ , so  ${}^t\underline{x}Q_i\underline{x} \in R_2$  if  $Q_i$  is a symmetric matrix.

The action of  $N \in \text{PGL}(5)$  is given by

$$N({}^t\underline{x}Q_1\underline{x}, {}^t\underline{x}Q_2\underline{x}, {}^t\underline{x}Q_3\underline{x})_j = {}^t\underline{x} {}^tN Q_j N \underline{x}, \quad 1 \leq j \leq 3.$$

The two actions commute and we can regard one as acting on the orbit space of the other.

In order to investigate properties of a smooth family of deformations of a canonical genus 5 curve  $C$  it is enough to consider the case in which the base scheme is the spectrum of  $A = \mathbb{K}[t_0, \dots, t_n]/\mathfrak{m}^2$ , where  $\mathfrak{m}$  is the maximal ideal generated by  $t_0, \dots, t_n$ , corresponding (as a point of  $\text{Spec } A$ ) to the curve  $C$ . Then the family is the scheme  $\mathcal{C} = \text{Proj } A[x_0, \dots, x_4]/(F_1, F_2, F_3)$ , where the coefficients of  $F_i$  depend linearly on the parameters  $t_i$ :

$$F_i = H_i + \sum_{j=0}^n t_j H_{ij},$$

where  $H_i, H_{ij} \in R_2$ . The  $n$  triples of quadrics  $(H_{1j}, H_{2j}, H_{3j})$  generate the linear subspace of  $\mathbb{A}^{45} = R_2 \times R_2 \times R_2$  tangent to the family  $\mathcal{C}$ .

We want to compare this linear space with the tangent space to  $\mathcal{M}_5$  at  $C$ . Our strategy is to work inside the tangent space to  $V$  at  $s$ : we construct a basis for all the trivial deformations using the fact that they are those given by the actions of  $\text{PGL}(5)$  and  $\text{GL}(3)$ , and then check how many of the above triples lie inside this linear space.

Around any point  $v = ({}^t\underline{x}Q_1\underline{x}, {}^t\underline{x}Q_2\underline{x}, {}^t\underline{x}Q_3\underline{x})$  in  $V$  the action of the two groups is linearised by the action of the corresponding Lie algebras, so a system of generators for the linear space tangent to the orbit passing through  $v$  is simply determined by applying a basis for the Lie algebra to it. The Lie algebra  $\mathfrak{gl}(3)$  is simply the whole space of three-by-three matrices

and its action is the same as the action of  $\mathrm{GL}(3)$  so we obtain a first set of trivial deformations given by the nine vectors

$$(H_1, 0, 0), (H_2, 0, 0), \dots, (0, 0, H_2), (0, 0, H_3).$$

The algebra  $\mathfrak{sl}(5)$  (which is the tangent space to  $\mathrm{PGL}(5)$ ) is the space of traceless  $5 \times 5$  matrices, and its action is determined as follows:

$$\begin{aligned} {}^t(N\underline{x})Q_i(N\underline{x}) &= {}^t\underline{x} {}^t(I + \varepsilon\Delta)Q_i(I + \varepsilon\Delta)\underline{x} \\ &= {}^t\underline{x}(Q_i + \varepsilon({}^t\Delta Q_i + Q_i\Delta))\underline{x} \\ &= {}^t\underline{x}Q_i\underline{x} + \varepsilon\underline{x}({}^t\Delta Q_i + Q_i\Delta)\underline{x}. \end{aligned}$$

Letting  $\Delta$  vary among a basis for  $\mathfrak{sl}(5)$  we get another set of trivial deformations given by the 24 vectors

$$({}^t\Delta H_1 + Q_1\Delta, {}^t\Delta H_2 + Q_2\Delta, {}^t\Delta H_3 + Q_3\Delta).$$

Now, given an  $n$ -dimensional family  $\mathcal{F}$  centred at  $C$ , we construct a matrix

$$M_{\mathcal{F}} := \begin{pmatrix} H_{11} & H_{21} & H_{31} \\ H_{12} & H_{22} & H_{32} \\ \vdots & \vdots & \vdots \\ H_{1n} & H_{2n} & H_{3n} \\ H_1 & 0 & 0 \\ H_2 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & H_2 \\ 0 & 0 & H_3 \\ D_{21}H_1 + H_1D_{12} & D_{21}H_2 + H_2D_{12} & D_{21}H_3 + H_3D_{12} \\ D_{31}H_1 + H_1D_{13} & D_{31}H_2 + H_2D_{13} & D_{31}H_3 + H_3D_{13} \\ \vdots & \vdots & \vdots \\ D_{55}H_1 + H_1D_{55} & D_{55}H_2 + H_2D_{55} & D_{55}H_3 + H_3D_{55} \end{pmatrix}$$

The first  $n$  rows are given by the family  $\mathcal{F}$ : they are tangent vectors at the central point  $s = (H_1, H_2, H_3)$ . The second set of 9 rows is given by the tangent vectors to the orbit of the  $\mathrm{GL}(3)$ -action, and the last 24 rows are the tangent vectors to the orbits of the  $\mathrm{PGL}(5)$ -action described above. We have chosen a vector space basis  $D_{ij}$  for  $\mathfrak{sl}(5)$ , for example  $D_{ij} = \delta_{ij}$  for  $i \neq j$  and  $D_{ii} = \delta_{11} - \delta_{ii}$  for  $1 < i \leq 4$ .

The linear space generated by the rows of  $M_{\mathcal{F}}$  is the span inside the tangent space to  $V$  of the three linear spaces tangent respectively to the given family and to each of the two orbits through  $s$ . To determine the dimension of this span we now need to compute the rank of  $M_{\mathcal{F}}$ .

**Proposition 4.2** *Let  $C$  be a smooth complete intersection of three linearly independent quadrics in  $\mathbb{P}^4$ , and let  $\mathcal{F}$  be an  $n$ -dimensional family of deformations of  $C$  as above. Suppose that  $n \geq 12$ . If the rank of  $M_{\mathcal{F}}$  is maximal then the Kodaira-Spencer map of  $\mathcal{F}$  at  $C$  is surjective.*

*Proof.* First observe that in the matrix  $M_{\mathcal{F}}$  there are 45 columns, and under our assumptions there are at least 45 rows. When the rank of the matrix  $M_{\mathcal{F}}$  is maximal the span of the three vector spaces, the two corresponding to trivial deformations and the one given by the family, is the whole of the tangent space to  $V$  at the point  $s$ . Thus we are guaranteed the existence of enough linearly independent deformations, namely 12, to fill the tangent space to  $\mathcal{M}_5$ .  $\square$

**Corollary 4.3** *The map  $\varrho: \mathcal{Q} \rightarrow \mathcal{R}_5$  is dominant.*

*Proof.* This is now a straightforward computation of the rank of  $M_{\mathcal{F}}$  in one particular case. We carried it out using *Macaulay*, with points defined over a finite field (we chose  $\mathbb{F}_{101}$ , for no special reason). This is enough because if the rank is generically maximal after reduction mod  $p$  it is also maximal in characteristic zero.

We chose the test point of  $\mathcal{Q}(\mathbb{F}_{101})$  given by

$$\begin{aligned} u_2 &= 19x_0^2 - 33x_0x_1 + 50x_1^2 - 13x_0x_2 + 50x_1x_2 - 15x_2^2 \\ u_3 &= -2x_0^2x_1 - 35x_0x_1^2 - 18x_0^2x_2 - 8x_0x_1x_2 - 36x_1^2x_2 - 4x_0x_2^2 + 45x_1x_2^2 \\ u_4 &= -38x_0^2x_1^2 - 32x_0^2x_1x_2 - 32x_0x_1^2x_2 - 6x_0^2x_2^2 - 38x_0x_1x_2^2 + 2x_1^2x_2^2. \end{aligned}$$

We arrived at this by first selecting six points  $P_0 = (0:0:0:1), \dots, P_3 = (1:0:0:0), P_4 = (1:1:1:1), P_5 = (1:2:3:4) \in \mathbb{P}^3$ , with ideals  $\mathfrak{p}_0, \dots, \mathfrak{p}_5$ , to be the prescribed nodes of a quartic and then choosing at random a quartic  $F \in \mathfrak{p}_0^2 \cap \dots \cap \mathfrak{p}_5^2$ . Then  $u_2, u_3$  and  $u_4$  are defined by  $F = u_2x_3^2 + u_3x_3 + u_4$ , and the 6-nodal quartic surface is given by  $F = 0$ .

Having chosen  $F$  at random one must check that it is suitably general, namely that  $X$  has no other singular points and that the singularity of  $X$  at each  $P_i$  is a simple node.

The 5-nodal plane sextic in this example is given by  $u_3^2 = u_2u_4$ , with nodes at  $(0:0:1), (0:1:0), (1:0:0), (1:1:1)$  and  $(1:2:3)$ . We construct the blowup of  $\mathbb{P}^2$  in these five points by considering the linear system of cubics passing through them and from this we can easily compute the canonical curve  $\tilde{C}$  as the intersection of three quadrics  $H_1, H_2$  and  $H_3$ . These at once give us the last 33 rows of  $M_{\mathcal{Q}}$ . To compute the first 13 rows one must know the family  $\mathcal{Q}$ , that is  $B_0$ , near  $X$ . We can obtain local coordinates on  $B_0$  from the coordinates on  $\mathbb{P}(I_4) \times \mathbb{P}^3$  by computing a Gröbner basis for the ideal of  $B_0$ . Then for each first-order deformation  $X_j$  corresponding to a local coordinate  $t_j$  we compute the quadrics defining the canonical curve

$\tilde{C}_j$  exactly as we did for  $\tilde{C}$ . These quadrics are  $H_i + t_j H_{ij}$  (with a correct choice of coordinates) and we have computed  $M_Q$ .  $\square$

## 5 Conclusions

We can now deduce our main result immediately from the results of Sections 3 or 4.

**Theorem 5.1** *The moduli space  $\mathcal{R}_5$  of étale double covers of curves of genus five is unirational.*

*Proof.* This follows from Corollary 3.3 or Corollary 4.3.  $\square$

Theorem 5.1 also provides a slightly different proof of a theorem of Clemens [7].

**Corollary 5.2**  *$\mathcal{A}_4$  is unirational.*

This follows from Theorem 5.1 because the Prym map  $p_5: \mathcal{R}_5 \rightarrow \mathcal{A}_4$  is dominant (see for instance [3]). The original proof of Clemens also starts from quartic double solids: Clemens exhibits the general principally polarised abelian 4-fold as an intermediate Jacobian rather than a Prym variety. Note that since  $\mathcal{R}_5$  has dimension 12 and  $\mathcal{A}_4$  has dimension 10, the dominance of the Intermediate Jacobian map from  $\mathcal{Q}$  to  $\mathcal{A}_4$  does *not* imply the dominance of the map  $\rho: \mathcal{Q} \rightarrow \mathcal{R}_5$ .

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