SECOND ORDER THETA DIVISORS ON PRYMS

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The Schottky problem is the problem of finding necessary and sufficient conditions for a principally polarized abelian variety (*ppav*) to be a product of jacobians of smooth curves.

Let (P, Ξ) be an *indecomposable* ppav of dimension $p \geq 4$ with Ξ a symmetric theta divisor on P. The elements of $\Gamma = H^0(P, 2\Xi)$ are symmetric, hence their multiplicities at the origin are always even. Let $\Gamma_0 \subset \Gamma$ be the subvector space of Γ of sections with multiplicity at least 2 at the origin and let $\Gamma_{00} \subset \Gamma$ be the subvector space of sections with multiplicity at least 4 at the origin. Also let $|2\Xi|_{00} \subset |2\Xi|_0 \subset |2\Xi|$ be the linear systems of divisors of zeros of elements of $\Gamma_{00} \subset \Gamma_0 \subset \Gamma$ respectively. It is well-known that the dimensions of Γ, Γ_0 and Γ_{00} are respectively 2^p, $2^p - 1$ and $2^p - 1 - \frac{p(p+1)}{2}$ (see [Ig] page 188 Lemma 11 and [GG] Proposition 1.1 page 618).

We have the linear map

$$\tau_4 : \Gamma_{00} \longrightarrow H^0(\mathbb{P}T_0P, \mathcal{O}_{\mathbb{P}T_0P}(4))$$

defined by sending a section $s \in \Gamma_{00}$ to the quartic term of its Taylor expansion at the origin. Let $\mathcal{Q}_{00}$ denote the linear subsystem of $|\mathcal{O}_{\mathbb{P}T_0P}(4)|$ which is the projectivization of the image of τ_4. Let V_{00} and $V_{inf,00}$ denote the base loci of $|2\Xi|_{00}$ and $\mathcal{Q}_{00}$ respectively. In [GG], van Geemen and van der Geer proposed a characterization of the locus of jacobians, made more precise by Donagi ([Do1] page 110), in the following form:

Conjecture 1. 1. *If $(P, \Xi) = (JC, \Theta)$ is the jacobian of a smooth curve C of genus p, then V_{00} is set-theoretically equal to the reduced surface $C - C := \{\mathcal{O}_C(s - t) : s, t \in C\}$.*
2. *If (P, Ξ) is (indecomposable and) not in the closure $\mathcal{J}_p$ of the locus of jacobians in the moduli space of ppav of dimension p, then $V_{00} = \{0\}$ set-theoretically.*

Beauville and Debarre proposed an infinitesimal version of Conjecture 1 ([BD]):

Conjecture 2. 1. *If $(P, \Xi) = (JC, \Theta)$ is the jacobian of a smooth curve C of genus p, then $V_{inf,00}$ is, set-theoretically, the canonical image κC of C in $|\omega_C|^* = \mathbb{P}T_0JC$ where ω_C is the dualizing sheaf of C (note that κC is the projectivized tangent cone to $C - C$ at 0).*
2. *If (P, Ξ) is (indecomposable and) not in $\mathcal{J}_p$, then $V_{inf,00}$ is empty.*

The first parts of Conjectures 1 and 2 have been proved (each with one well-determined exception) by Welters [W1], the author [Iz1] and Beauville and Debarre [BD]. In [Iz1], we also determined the scheme-structures of the base loci for jacobians. In [Iz2], Theorem 4 page 95, we proved the second parts of Conjectures 1 and 2 in the case $p = 4$. Beauville, Debarre, Donagi and van der Geer proved part 2 of Conjecture 1 for intermediate jacobians of cubic threefolds and the Prym varieties of "even" étale double covers of smooth plane curves (see [BDDG]). Beauville and Debarre proved parts 2 of Conjectures 1 and 2 for certain ppav isogenous to a product of p elliptic curves (see [BD] pages 35-38). By semi-continuity, Beauville and Debarre then deduce from their result that for a *general* ppav V_{00} is finite and $V_{inf,00}$ is empty.

Let $\mathcal{A}_p$ be the moduli space of ppav of dimension p. In [Iz2], we proved the second parts of Conjectures 1 and 2 for $p = 4$ by using the fact (proved in [Iz2], Theorem 3.3 page 111) that an element of $\mathcal{A}_4 \setminus \mathcal{J}_4$ is always the Prym variety of an étale double cover of a *smooth* curve of genus 5. From now on we will suppose that (P, Ξ) is the Prym variety of an étale double cover of smooth curves $\pi : \widetilde{C} \to C$ with C *non-hyperelliptic* of genus $g = p+1$ (then (P, Ξ) is automatically indecomposable, see [M2] page 344, Theorem(d)). There is a natural analogue of the surface $C - C$ for a Prym variety, namely, the reduced surface

$$\Sigma := \Sigma(\pi : \widetilde{C} \to C) := \{\mathcal{O}_{\widetilde{C}}(s + t - \sigma s - \sigma t) : s, t \in \widetilde{C}\} \subset P \subset J\widetilde{C},$$

where $\sigma : \widetilde{C} \to \widetilde{C}$ is the involution of the cover $\pi : \widetilde{C} \to C$. Let $\epsilon : \widetilde{P} \to P$ be the blow up of P at 0 with exceptional divisor $\mathcal{E}$ and let $\widetilde{\Sigma}$ be the proper transform of Σ in $\widetilde{P}$. Let L be the linear system $|\epsilon^*(2\Xi) - 4\mathcal{E}|$ on $\widetilde{P}$. When $g = 5$ there is an involution λ acting on the moduli space of admissible double covers of stable curves of genus 5 such that a double cover $\widetilde{C} \to C$ and $\lambda(\widetilde{C} \to C)$ have the same Prym variety (see [Do2] page 100 and [Iz2] pages 119 and 126). Furthermore, for any fixed $(P, \Xi) \in \mathcal{A}_4 \setminus \mathcal{J}_4$, there is an étale double cover $\widetilde{C} \to C$ of a smooth curve C such that $(\widetilde{C}_\lambda \to C_\lambda) := \lambda(\widetilde{C} \to C)$ is also an étale double cover of a smooth curve C_λ (see [Iz2] page 136). In such a case, put $\Sigma_\lambda := \Sigma(\widetilde{C}_\lambda \to C_\lambda)$ and let $\widetilde{\Sigma}_\lambda$ be the proper transform of Σ_λ in $\widetilde{P}$. With these hypotheses, we proved in [Iz2] that (recall $g - 1 = p = 4$)

- there is exactly a pencil of elements of L containing $\widetilde{\Sigma}$ (see [Iz2], 5.7 page 134 and 6.23 page 148),
- the base locus of this pencil is equal to $\widetilde{\Sigma} \cup \widetilde{\Sigma}_\lambda$ as a set, as a scheme if $(\widetilde{C} \to C) \neq (\widetilde{C}_\lambda \to C_\lambda)$ (follows from [Iz2], 5.7 page 134),
- the base locus of the restriction $L|_{\widetilde{\Sigma}}$ (and, similarly, $L|_{\widetilde{\Sigma}_\lambda}$) is empty ([Iz2] pages 139 and 146-147).

We generalize this third result to higher-dimensional Prym varieties and calculate the dimension of the linear subsystem of L consisting of elements containing $\widetilde{\Sigma}$:

In Section 4 we prove

Theorem 3. *If C is non-trigonal of genus ≥ 5, then the base locus of $L|_{\widetilde{\Sigma}}$ is empty.*

For C trigonal, the support of the base locus of $L|_{\widetilde{\Sigma}}$ is determined in Proposition 4.8 below.

To prove the Theorem, we use divisors in the linear system $|2\Xi|_{00}$ which are obtained as intersections with $P \subset J\widetilde{C}$ of translates of the theta divisor $\widetilde{\Theta}$ of $J\widetilde{C}$ (see Section 3 below). To our knowledge such divisors have not been used before in the litterature.

The Prym-canonical curve χC is the image of C in $|\omega_C \otimes \alpha|^*$ by the natural morphism $C \to |\omega_C \otimes \alpha|^*$ where α is the square-trivial invertible sheaf associated to the double cover $\pi : \widetilde{C} \to C$. Under the natural isomorphism $\mathbb{P}T_0 P \cong |\omega_C \otimes \alpha|^*$, the curve χC is the tangent cone to Σ at 0. Therefore the Theorem implies

Corollary 4. *Suppose that C is not trigonal, then*

1. *the only base point of $|2\Xi|_{00}$ on Σ is 0,*
2. *the linear system $\mathcal{Q}_{00}$ has no base points on χC.*

Our approach leads us to consider the vector space $\Gamma'_{00} := \{s \in \Gamma_{00} : s|_\Sigma = 0\}$ with projectivization $|2\Xi|'_{00} := \{D \in |2\Xi|_{00} : D \supset \Sigma\}$. Then $|2\Xi|'_{00}$ can be identified with the linear subsystem of elements of L containing $\widetilde{\Sigma}$. Let $\mathcal{Q}'_{00}$ be the linear subsystem of $\mathcal{Q}_{00}$ consisting of quartic tangent cones at 0 to elements of $|2\Xi|'_{00}$. On a Prym variety, second order theta divisors which contain Σ can be thought of as natural generalizations of 2Θ-divisors on JC containing $C - C$. It is well-known (see [F] Theorem 2.5 page 120 and [GG] page 625 or [W1] Proposition 4.8 page 18) that, on a jacobian (JC, Θ), a 2Θ-divisor contains $C - C$ if and only if it has multiplicity at least 4 at the origin. This does not generalize to Prym varities: Corollary 4 implies that $|2\Xi|'_{00}$ and $\mathcal{Q}'_{00}$ are proper linear subsystems of $|2\Xi|_{00}$ and $\mathcal{Q}_{00}$ respectively. More precisely, we prove (see Section 4):

Proposition 5. *The dimension of* Γ'_{00} *is* $2^p - 2 - p(p-1)$*. The codimension of* $\mathcal{Q}'_{00}$ *in* $\mathcal{Q}_{00}$ *is at least* $g-3$*.*

Note that when $p = 4$, we have another proof of the result of [Iz2] (page 148) saying that the dimension of $|2\Xi|'_{00}$ is 1. We pose

Conjecture 6. *Suppose that* (P, Ξ) *is not a jacobian. Then*

1. *the base locus* V'_{00} *of* $|2\Xi|'_{00}$ *is* Σ *as a set if* $p \geq 6$*,*
2. *the base locus* $V'_{inf,00}$ *of* $\mathcal{Q}'_{00}$ *is* χC *as a set if* $p \geq 6$*.*

By Corollary 4, this conjecture implies the second parts of Conjectures 1 and 2 for Prym varieties. We have the following evidence for this conjecture:

Results of Welters and Debarre ([W2] and [De], see Section 1 below) imply

Proposition 7. *The base locus* V'_{00} *is the set-theoretical union of* Σ *and, possibly, some curves and points for a general Prym variety of dimension* $p \geq 16$*.*

Combined with Corollary 4 this implies:

Corollary 8. *If* (P, Ξ) *is a general Prym variety of dimension* ≥ 16*, then* V_{00} *has dimension* ≤ 1*.*

Results of Debarre imply (see Section 1)

Proposition 9. *For a general Prym variety of dimension* $p \geq 8$*, the base locus* $V'_{inf,00}$ *is set-theoretically equal to* χC*.*

Combined with Corollary 4 this implies (see Section 1 below)

Corollary 10. *Part 2 of Conjecture 2 is true for general Prym varieties of dimension* $p = g-1 \geq 8$*.*

We explain why we make Conjecture 6 only for $p \geq 6$ and only set-theoretically. Let $\mathcal{R}_g$ be the space parametrizing étale double covers $\widetilde{C} \to C$ where C is a smooth (non-hyperelliptic) curve of genus $g = p+1$. The Prym map is the morphism $\mathcal{R}_g \to \mathcal{A}_p$ which to a double cover $\widetilde{C} \to C$ associates its Prym variety. Recall (see above) that in case $p = 4$, we proved in [Iz2] (5.7 page 134) that $V'_{00} = \Sigma \cup \Sigma_\lambda$. Since the Prym map is generically injective for $g \geq 7$ (see [FS]), we expect that $V'_{00} = \Sigma$ as sets. Now an argument analogous to [Iz1] (2.9) page 196 shows that, if $V'_{00} = \Sigma$ as sets, then V'_{00} is not reduced at 0 and hence is not equal to Σ as a scheme. If the morphism $\rho : \widetilde{C}^{(2)} \to \Sigma$, where $\widetilde{C}^{(2)}$ is the second symmetric power of $\widetilde{C}$, is birational, then, by analogy with the case of jacobians (see [Iz1]), we can expect V'_{00} to be reduced at the generic point of Σ. If, on the other hand, the morphism ρ is not birational, then, by a standard semi-continuity argument, the scheme V'_{00} is not reduced anywhere on Σ. Note that, using a refinement of a theorem of Martens by Mumford ([ACGH] Theorem 5.2 page 193), one can easily see that if $\widetilde{C}$ is neither bielliptic, trigonal, nor a smooth plane quintic, then ρ is birational. Similarly, we can expect $V'_{inf,00}$ to be equal to χC as schemes if $C \cong \chi C$ but $V'_{inf,00}$ will not be equal to χC as a scheme if the morphism $C \to \chi C$ is not (at least) birational.

On the other hand, Donagi and Smith proved (see [DS]) that the generic fibers of the Prym map have cardinality 27 when $g = 6$ and Donagi proved (see [Do2] Theorem 4.1 page 90) that the Galois group of the Prym map is isomorphic to the Galois group of the 27 lines on a cubic surface. So there could be nontrivial automorphisms acting in the fibers of the Prym map. If there is an automorphism $\mu : \mathcal{R}_6 \to \mathcal{R}_6$ acting in the fibers of the Prym map, then, as in the case $p = 4$, the base locus V'_{00} could be the union of $\Sigma = \Sigma(\widetilde{C} \to C)$ and the surfaces $\Sigma(\mu(\widetilde{C} \to C)), \Sigma(\mu^2(\widetilde{C} \to C))$, etc.

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Notation and conventions

All varieties and schemes are over the field $\mathbb{C}$ of complex numbers. All ppav are indecomposable and all curves are smooth, complete, irreducible and non-hyperelliptic.

For any section s of an invertible sheaf $\mathcal{L}$ on a variety X, denote by $Z(s)$ the divisor of zeros of s. Let $h^i(X, \mathcal{L})$ denote the dimension of $H^i(X, \mathcal{L})$.

For a divisor D on C (resp. $\widetilde{C}$), denote by $\langle D \rangle$ the span of D in the canonical space of C (resp. $\widetilde{C}$).

For any subset Y of a group G and any element $a \in G$, denote by Y_a the translate of Y by a.

1. Preliminaries

Let $\pi : \widetilde{C} \to C$ be an étale double cover of a smooth (non-hyperelliptic) curve C of genus g. Let α be the point of order 2 in $Pic^0 C$ associated to the double cover π so that we have $\pi_* \mathcal{O}_{\widetilde{C}} \cong \mathcal{O}_C \oplus \alpha$. Choose an element β of $Pic^0 C$ such that $\beta^{\otimes 2} \cong \alpha$ and a theta-characteristic κ on C such that $h^0(C, \kappa)$ and $h^0(\widetilde{C}, \pi^*(\kappa \otimes \beta))$ are even. Symmetric principal polarizations on $J\widetilde{C} = Pic^0 \widetilde{C}$ and $JC = Pic^0 C$ can be defined as the reduced divisors $\widetilde{\Theta} := \widetilde{\Theta}'_{(\kappa\otimes\beta)^{-1}}$ and $\Theta := \Theta'_{\kappa^{-1}}$ where

$$\widetilde{\Theta}' := \{D \in Pic^{2g-2} \widetilde{C} : h^0(\widetilde{C}, D) > 0\}$$

$$\Theta' := \{D \in Pic^{g-1} C : h^0(C, D) > 0\} \, .$$

With these definitions, the inverse image of $\widetilde{\Theta}$ by the morphism $\pi^* : JC \to J\widetilde{C}$ is the divisor $\Theta_\beta + \Theta_{\beta^{-1}}$. The Prym variety (P, Ξ) of the double cover $\pi : \widetilde{C} \to C$ is defined by the reduced varieties $P := P'_{(\kappa\otimes\beta)^{-1}}$ and $\Xi := \Xi'_{(\kappa\otimes\beta)^{-1}}$ with

$$P' := \{E \in Pic^{2g-2} \widetilde{C} : Nm(E) \cong \omega_C, h^0(\widetilde{C}, E) \equiv 0 \bmod 2\}$$

and

$$\Xi' := \{E \in P' : h^0(\widetilde{C}, E) > 0\},$$

where $Nm : Pic\widetilde{C} \to PicC$ is the Norm map (see [M2] pages 331-333 and 340-342). As divisors we have $2\Xi = P.\widetilde{\Theta}$.

For any $E \in P'$, since $Nm(E) \cong \omega_C$, we have $\omega_{\widetilde{C}} \otimes E^{-1} \cong \sigma^* E$. By the theorem of the square, the divisor $\Xi'_{E^{-1}} + \Xi'_{\sigma^* E^{-1}}$ is in the linear system $|2\Xi|$ and, by Wirtinger Duality (see [M2] pages 335-336), such divisors span $|2\Xi|$. Furthermore, if $E \in \Xi'$, then $\Xi'_{E^{-1}} + \Xi'_{\sigma^* E^{-1}}$ is in $|2\Xi|_0$ and such divisors span $|2\Xi|_0$ (by Wirtinger duality, the span of such divisors is the span of $\phi(\Xi)$ where $\phi : P \to |2\Xi|^*$ is the natural morphism, this span is a hyperplane in $|2\Xi|^*$ which can therefore be identified with $|2\Xi|_0$ by Wirtinger duality, also see [W1] page 18).

We now explain how Proposition 7 follows from results of Welters and Debarre and how Proposition 9 follows from results of Debarre. In this paragraph only, suppose that (P, Ξ) is a *general* Prym variety. Then E is an element of the singular locus $Sing(\Xi')$ of Ξ' if and only if $h^0(\widetilde{C}, E) \geq 4$ ([W2] page 168). Therefore, for every $E \in Sing(\Xi')$ and $(p, q) \in \widetilde{C}^2$, we have $h^0(\widetilde{C}, E \otimes \mathcal{O}_{\widetilde{C}}(p + q - \sigma p - \sigma q)) > 0$. Hence $\Sigma \subset \Xi'_{E^{-1}}$ and $\Xi'_{E^{-1}} + \Xi'_{\sigma^* E^{-1}}$ is in $|2\Xi|'_{00}$. Since Ξ is symmetric, the tangent cones at 0 to $\Xi'_{E^{-1}}$ and $\Xi'_{\sigma^* E^{-1}}$ are equal. It follows from these facts and the irreducibility of $Sing(\Xi')$ for $p \geq 6$ (see [De] Théorème 1.1 page 114) that V'_{00} is set-theoretically contained in $\cap_{E \in Sing(\Xi')} \Xi'_{E^{-1}}$ and $V'_{inf,00}$ is set-theoretically contained in the intersection of the tangent cones at 0 to $\Xi'_{E^{-1}}$ for E a point of multiplicity 2 on Ξ'. This latter can be rephrased as: $V'_{inf,00}$ is set-theoretically contained in the intersection of the tangent cones to Ξ' at its points of multiplicity 2. It is proved in [W2] (Theorem 2.6 page 169) that, for $p \geq 16$, the intersection $\cap_{E \in Sing(\Xi')} \Xi'_{E^{-1}}$ is the set-theoretical union of Σ and, possibly, some curves and points. This proves Proposition 7. By [De] Théorème 1.1 page 114, the tangent cones to Ξ' at its points of multiplicity 2 generate the space of quadrics containing χC for $p \geq 6$ and, for $p \geq 8$ (see [De] Corollaire 2.3 page 129, also see [L] and [LS]), the Prym-canonical curve χC is cut out by quadrics. This proves Proposition 9.

2. PULL-BACKS OF DIVISORS TO $\widetilde{C}^2$

Let $\rho : \widetilde{C}^2 \to P \subset J\widetilde{C}$ be the morphism

$$\rho : (s,t) \longmapsto [s,t] := \mathcal{O}_{\widetilde{C}}(s + t - \sigma s - \sigma t)$$

so that Σ is the image of $\widetilde{C}^2$ by ρ. The morphism ρ lifts to a morphism $\widetilde{\rho} : \widetilde{C}^2 \twoheadrightarrow \widetilde{\Sigma} \subset \widetilde{P}$. We have

Lemma 2.1. *Let $E \in \Xi'$ be such that $h^0(\widetilde{C}, E) = 2$ and let B be the base divisor of $|E|$. Then the inverse image of $\Xi'_{E^{-1}}$ in $\widetilde{C}^2$ is the divisor*

$$\rho^* \Xi'_{E^{-1}} = D_{\sigma^* E} + \Delta'$$

with $D_E = D_{E \otimes \mathcal{O}_{\widetilde{C}}(-B)} + p_1^* B + p_2^* B$ where $p_i : \widetilde{C}^2 \to \widetilde{C}$ is the projection onto the i-th factor, the divisor $D_{E \otimes \mathcal{O}_{\widetilde{C}}(-B)}$ is reduced and equal to

$$D_{E \otimes \mathcal{O}_{\widetilde{C}}(-B)} := \{(p,q) : h^0(\widetilde{C}, E \otimes \mathcal{O}_{\widetilde{C}}(-B - p - q)) > 0\}$$

and Δ' is the "pseudo-diagonal" of $\widetilde{C}^2$, i.e., the reduced curve $\Delta' := \{(p, \sigma p) \in \widetilde{C}^2\} = \rho^{-1}(0)$.

Furthermore, the divisor $\rho^* \Xi'_{E^{-1}}$ is in the linear system $|p_1^* \sigma^* E \otimes p_2^* \sigma^* E \otimes \mathcal{O}_{\widetilde{C}}(-\Delta + \Delta')|$.

Proof : We have the equality of sets

$$\rho^* \Xi'_{E^{-1}} = \{(p,q) : E \otimes [p,q] \in \Xi'\} = \{(p,q) : h^0(\widetilde{C}, E \otimes [p,q]) > 0\} \ .$$

This first implies that $\rho^* \Xi'_{E^{-1}}$ is a divisor: for general points p and q in $\widetilde{C}$, we have $h^0(\widetilde{C}, E \otimes \mathcal{O}_{\widetilde{C}}(p + q)) = 2$ and $h^0(\widetilde{C}, E \otimes \mathcal{O}_{\widetilde{C}}(-\sigma p - \sigma q)) = 0$. Therefore p and q are base points for $|E \otimes \mathcal{O}_{\widetilde{C}}(p+q)|$ and $h^0(\widetilde{C}, E \otimes \mathcal{O}_{\widetilde{C}}(p + q - \sigma p - \sigma q)) = 0$.

Secondly, if $[p,q] = \mathcal{O}_{\widetilde{C}}$, i.e., $p = \sigma q$, or if $h^0(\widetilde{C}, E \otimes \mathcal{O}_{\widetilde{C}}(-B - \sigma p - \sigma q)) > 0$, then $[p,q] \in \rho^* \Xi'_{E^{-1}}$. So $\rho^* \Xi'_{E^{-1}} - D_{\sigma^* E \otimes \mathcal{O}_{\widetilde{C}}(-\sigma B)} - \Delta'$ is effective.

Now, if B is zero, an easy degree computation on the fibers of $\widetilde{C}^2$ over $\widetilde{C}$ by the two projections shows that $\rho^* \Xi'_{E^{-1}} = D_{\sigma^* E} + \Delta'$ as divisors.

Restricting to fibers of p_1 and p_2 and using the See-Saw Theorem, we see that $D_{\sigma^* E \otimes \mathcal{O}_{\widetilde{C}}(-\sigma B)}$ is in the linear system $|p_1^*(\sigma^* E \otimes \mathcal{O}_{\widetilde{C}}(-\sigma B)) \otimes p_2^*(\sigma^* E \otimes \mathcal{O}_{\widetilde{C}}(-\sigma B)) \otimes \mathcal{O}_{\widetilde{C}}(-\Delta)|$ where Δ is the diagonal of $\widetilde{C}^2$.

Therefore $\rho^* \Xi'_{E^{-1}}$ is in the linear system $|p_1^* \sigma^* E \otimes p_2^* \sigma^* E \otimes \mathcal{O}_{\widetilde{C}}(-\Delta + \Delta')|$ when B is zero, and, by continuity, also when B is non-zero.

So $\rho^* \Xi'_{E^{-1}} - D_{\sigma^* E \otimes \mathcal{O}_{\widetilde{C}}(-\sigma B)} - \Delta'$ is linearly equivalent to $p_1^*(\sigma B) + p_2^*(\sigma B)$ and is effective. Since, by the Künneth isomorphism, the linear system $|p_1^*(\sigma B) + p_2^*(\sigma B)|$ has only one element, it follows that $\rho^* \Xi'_{E^{-1}} - D_{\sigma^* E \otimes \mathcal{O}_{\widetilde{C}}(-\sigma B)} - \Delta' = p_1^*(\sigma B) + p_2^*(\sigma B)$ as divisors. ☐

Remark 2.2. *Suppose that* E *as above can be written as* $E = \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(B)$ *where* M *is an invertible sheaf on* C *with* $h^0(C, M) = 2$. *Then* $D_{\sigma^* E} = \Delta' + D'$ *for some effective divisor* D'. *Therefore* $\rho^* \Xi'_{E^{-1}} = D' + 2\Delta'$ *which agrees with the fact that in such a case* $E \in Sing(\Xi')$ *so that* $\Xi'_{E^{-1}}$ *is singular at* 0 *(see* [M2] *pages 342-343).*

Let $\omega_{\widetilde{C}^2}$ be the canonical sheaf of $\widetilde{C}^2$. Then $\omega_{\widetilde{C}^2} \cong p_1^* \omega_{\widetilde{C}} \otimes p_2^* \omega_{\widetilde{C}}$ and we have Künneth's isomorphism $H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2}) \cong H^0(\widetilde{C}, \omega_{\widetilde{C}})^{\otimes 2}$. Let $I_2(\widetilde{C}) \subset S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}}) \subset H^0(\widetilde{C}, \omega_{\widetilde{C}})^{\otimes 2} = H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2})$ be the vector space of quadratic forms vanishing on the canonical image $\kappa \widetilde{C}$ of $\widetilde{C}$. Fix an embedding $H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta)) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2})$ obtained by multiplication by a nonzero global section of $\mathcal{O}_{\widetilde{C}^2}(2\Delta)$ (note that $h^0(\widetilde{C}^2, \mathcal{O}_{\widetilde{C}^2}(2\Delta)) = 1$ because Δ has negative self-intersection, hence any two such embeddings differ by multiplication by a constant). Then it is easily seen that

$$I_2(\widetilde{C}) = S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}}) \cap H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta)) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2})\,.$$

Similarly fix embeddings $H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta - 2\Delta')) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta)) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta + 2\Delta'))$. For $E \in \Xi'$ such that $h^0(\widetilde{C}, E) = 2$, it is well-known (see, e.g., [ACGH] page 261) that $q_E := \cup_{D1 \in |E|} \langle D \rangle = \cup_{D \in |\sigma^* E|} \langle D \rangle$ is a quadric of rank ≤ 4 whose ruling(s) cut(s) the divisors of the moving parts of $|E|$ and $|\sigma^* E|$ on $\widetilde{C}$. We need the following

Lemma 2.3. 1. *We have*

$$\rho^* \mathcal{O}_P(2\Xi) \cong \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta + 2\Delta'),$$

$$\rho^* \Gamma_0 \subset I_2(\widetilde{C}) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta)) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta + 2\Delta'))\,,$$

and

$$\rho^* \Gamma_{00} \subset I_2(\widetilde{C}) \cap H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta - 2\Delta')) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta + 2\Delta')).$$

2. *For any* $f \in I_2(\widetilde{C}) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta + 2\Delta'))$, *let* $q(f)$ *be the quadric in* $|\omega_{\widetilde{C}}|^*$ *with equation* f. *Then, in* $\widetilde{C}^2$, *the zero locus of* $f \in H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta + 2\Delta'))$ *is*

$$Z(f) = Z_{q(f)} + 2\Delta'$$

where $Z_{q(f)}$ *is a divisor with support*

$$\{(p, q) \in \widetilde{C}^2 : \langle p + q \rangle \subset q(f)\}.$$

For any $s \in \Gamma_0$, *put* $q(s) = q(\rho^* s)$. *For a general* $s \in \Gamma_0$, *the divisor* $Z_{q(s)}$ *is reduced. In particular, for* f *general, the divisor* $Z_{q(f)}$ *is reduced. If* $Z(s) = \Xi'_{E^{-1}} + \Xi'_{\sigma^* E^{-1}}$ *for some* $E \in \Xi'$ *such that* $h^0(\widetilde{C}, E) = 2$, *then* $q(s) = q_E := \cup_{D \in |E|} \langle D \rangle$.

3. *If* $s \in \Gamma_0 \setminus \Gamma_{00}$, *then*

$$q(s) \cap (\mathbb{P} T_0 P = |\omega_C \otimes \alpha|^*) \subset \mathbb{P} T_0 J \widetilde{C} = |\omega_{\widetilde{C}}|^*$$

is the projectivized tangent cone $\tau_{Z(s)}$ *to* $Z(s) \subset P$ *at* 0.

4. *For any* $s \in \Gamma_0$, *the multiplicity of* $\rho^* s$ *at the generic point of* Δ' *is even* ≥ 2 *and if* $\rho^* s$ *vanishes on* Δ' *with multiplicity* ≥ 4, *then either* $s \in \Gamma_{00}$ *or* $\tau_{Z(s)}$ *contains the Prym-canonical curve* χC.

Proof : 1. Let E be an invertible sheaf of degree $2g - 2$ on $\widetilde{C}$ such that $h^0(\widetilde{C}, E) = 2$ and $E \otimes \sigma^* E \cong \omega_{\widetilde{C}}$. Then, by Lemma 2.1, we have $\rho^*(\Xi'_{E^{-1}} + \Xi'_{\sigma^* E^{-1}}) = D_{\sigma^* E} + D_E + 2\Delta' \in |p_1^* \sigma^* E \otimes p_2^* \sigma^* E \otimes p_1^* E \otimes p_2^* E \otimes \mathcal{O}_{\widetilde{C}}(-2\Delta + 2\Delta')| = |\omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}}(-2\Delta + 2\Delta')|$. This proves the first assertion.
Now, let s_1 and s_2 be two general sections of E. Then

$$s_1 \otimes s_2 - s_2 \otimes s_1 \in \Lambda^2 H^0(\widetilde{C}, E) \subset H^0(\widetilde{C}, E)^{\otimes 2}$$

and, as in the the proof of Lemma 2.1, it is easily seen that $Z(s_1 \otimes s_2 - s_2 \otimes s_1) = D_E + \Delta$.

From the natural map

$$\psi_E : H^0(\widetilde{C}, E) \otimes H^0(\widetilde{C}, \sigma^* E) \longrightarrow H^0(\widetilde{C}, \omega_{\widetilde{C}})$$

we obtain the map

$$\begin{array}{rcl} H^0(\widetilde{C}, E)^{\otimes 2} \otimes H^0(\widetilde{C}, \sigma^* E)^{\otimes 2} & \longrightarrow & H^0(\widetilde{C}, \omega_{\widetilde{C}})^{\otimes 2} \cong H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2}) \\ t_1 \otimes t_2 \otimes \sigma^* u_1 \otimes \sigma^* u_2 & \longmapsto & \psi_E(t_1 \otimes \sigma^* u_1) \otimes \psi_E(t_2 \otimes \sigma^* u_2) \end{array}$$

which induces the map

$$\phi_E : \Lambda^2 H^0(\widetilde{C}, E) \otimes \Lambda^2 H^0(\widetilde{C}, \sigma^* E) \longrightarrow S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}})$$

Put

$$t := (s_1 \otimes s_2 - s_2 \otimes s_1) \otimes (\sigma^* s_1 \otimes \sigma^* s_2 - \sigma^* s_2 \otimes \sigma^* s_1)$$

then $Z(\phi_E(t))$ is equal to $D_E + \Delta + D_{\sigma^* E} + \Delta$.

If $s \in \Gamma_0$ is such that $Z(s) = \Xi'_{E^{-1}} + \Xi'_{\sigma^* E^{-1}}$, then, by Lemma 2.1, we have $Z(\rho^* s) = D_E + D_{\sigma^* E} + 2\Delta'$. So $Z(\rho^* s) - 2\Delta' = Z(\phi_E(t)) - 2\Delta$ and the section $\rho^* s$ of $\rho^* \mathcal{O}_P(2\Xi) \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta') \cong \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta)$ is a nonzero constant multiple of $\phi_E(t) \in S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}}) \cap H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta)) = I_2(\widetilde{C}) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta))$. Since such s generate Γ_0, this proves that $\rho^* \Gamma_0 \subset I_2(\widetilde{C})$. The rest of part 1 easily follows now.

2. First think of f as an element of $S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}}) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2})$. We can write $f = \sum_{i=1}^r p_1^* \omega_i \otimes p_2^* \omega_i$ for some $\omega_i \in H^0(\widetilde{C}, \omega_{\widetilde{C}})$. Then $Z(f) = \{(p,q) : \sum_{i=1}^r \omega_i(p) \omega_i(q) = 0\}$ as sets. Since $f \in I_2(\widetilde{C})$, we have $\sum_{i=1}^r \omega_i(p)^2 = 0$ for every $p \in \widetilde{C}$. Therefore $Z(f) \supset \Delta$. Furthermore, for any two distinct points p and q of C, the equations $\sum_{i=1}^r \omega_i(p)^2 = \sum_{i=1}^r \omega_i(p)\omega_i(q) = \sum_{i=1}^r \omega_i(q)^2 = 0$ mean that the line $\langle p + q \rangle$ is in $q(f)$. Therefore $Z(f) = \{(p,q) : \langle p+q \rangle \subset q(f)\} \cup \Delta$ as sets. Since $f \in S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}})$, it vanishes with even multiplicity on Δ. So $Z(f) = Z_{q(f)} + 2\Delta$ where $Z_{q(f)}$ has support $\{(p,q) : \langle p+q \rangle \subset q(f)\}$. Finally, if we think of f as a section of $\omega_{\widetilde{C}^2} \otimes \mathcal{O}_{\widetilde{C}^2}(-2\Delta + 2\Delta')$, then $Z(f) = Z_{q(f)} + 2\Delta'$.

Let s and E be as in 1. With the notation of 1, let X_1, X_2, X_3, X_4 be the images of, respectively,

$$s_1 \otimes \sigma^* s_1, s_2 \otimes \sigma^* s_1, s_1 \otimes \sigma^* s_2, s_2 \otimes \sigma^* s_2$$

by the map ψ_E. Then $\phi_E(t) = X_1 X_4 - X_2 X_3$. By, e.g. [ACGH] page 261, the polynomial $X_1 X_4 - X_2 X_3$ is an equation for q_E. Therefore, since $\rho^* s$ is a constant nonzero multiple of $\phi_E(t)$, we have $q(s) = q_E$. Hence

$$D_E + D_{\sigma^* E} = Z_{q_E} = Z_{q(s)}.$$

The subvariety of $\widetilde{C}^{(2g-2)}$ parametrizing divisors D such that $\mathcal{O}_{\widetilde{C}}(D) \in \Xi'$ maps dominantly to $|\omega_C|$ via $D \mapsto \pi_* D$. Therefore, for E general in Ξ', the linear systems $|E|$ and $|\sigma^* E|$ contain reduced divisors and D_E and $D_{\sigma^* E}$ are reduced. Furthermore, the base loci B and σB have no points in common (because $\pi_* |E|$ contains reduced divisors). Therefore, if the moving parts of $|E|$ and $|\sigma^* E|$ are distinct, the divisors D_E and $D_{\sigma^* E}$ have no common components and their sum is reduced. If this is not the case, then $E = \pi^* M \otimes \mathcal{O}_{\widetilde{C}}(B)$ for some effective line bundle M on C. Counting dimensions, we see that this does not happen for a general $E \in \Xi'$. Thus, if $Z(s) = \Xi'_{E^{-1}} + \Xi'_{\sigma^* E^{-1}}$ for $E \in \Xi'$ general, then $Z_{q(s)}$ is a reduced divisor. Hence $Z_{q(s)}$ is reduced for general $s \in \Gamma_0$.

3. When E is a smooth point of Ξ', the intersection $q_E \cap \mathbb{P} T_0 P = 2 \mathbb{P} T_E \Xi'$ is the projectivized tangent cone at 0 to $\Xi'_{E^{-1}} + \Xi'_{\sigma^* E^{-1}}$ (see [M2] pages 342-343). Now 3 follows by linearity.

4. This immediately follows from the facts that $\rho^* s \in I_2(\widetilde{C}) \subset S^2 H^0(\widetilde{C}, \omega_{\widetilde{C}}) \subset H^0(\widetilde{C}^2, \omega_{\widetilde{C}^2})$ and that χC is the tangent cone at 0 to Σ.

□

3. More divisors in $|2\Xi|_{00}$

For any $M \in Pic^{g-1}C$ we have

Proposition 3.1. *The divisor $P.\widetilde{\Theta}'_{\pi^*M^{-1}}$ is in the linear system $|2\Xi|$. It is in $|2\Xi|_0$ if $h^0(C,M)$ is positive.*

Proof : We first prove that all the divisors $P.\widetilde{\Theta}'_{\pi^*M^{-1}}$ are linearly equivalent as M varies in $Pic^{g-1}C$. Let $\psi : JC \to Pic^0P$ be the morphism of abelian varieties which sends $M \otimes \kappa^{-1} \otimes \beta^{-1}$ to $\mathcal{O}_P(P.\widetilde{\Theta}'_{\pi^*M^{-1}} - P.\widetilde{\Theta}) \in Pic^0P$. Then the map ψ is the dual of the zero map $P \hookrightarrow J\widetilde{C} \overset{Nm}{\to} JC$. Hence the image of ψ is $\mathcal{O}_P$. Since $P.\widetilde{\Theta} = 2\Xi$, all the divisors $P.\widetilde{\Theta}'_{\pi^*M^{-1}}$ are linearly equivalent to 2Ξ. The second assertion is now immediate. ☐

Proposition 3.2. *The divisor $P.\widetilde{\Theta}'_{\pi^*M^{-1}}$ is an element of $|2\Xi|_{00}$ if $h^0(C,M) \geq 2$.*

Proof : If $h^0(\widetilde{C},\pi^*M) > 2$, then $\widetilde{\Theta}'_{\pi^*M^{-1}}$ has multiplicity at least three at the origin whence so has its restriction $P.\widetilde{\Theta}'_{\pi^*M^{-1}}$. Since this multiplicity is even, it is at least 4.

Suppose therefore that $h^0(\widetilde{C},\pi^*M) = 2$. By, e.g. [ACGH] page 261, the tangent cone to $\widetilde{\Theta}'_{\pi^*M^{-1}}$ at 0 is the quadric $q_{\pi^*M} := \cup_{\delta\in|\pi^*M|}\langle\delta\rangle$ in $|\omega_{\widetilde{C}}|^* = \mathbb{P}T_0J\widetilde{C}$. Let π also denote the projection $\mathbb{P}T_0J\widetilde{C} = |\omega_{\widetilde{C}}|^* \to |\omega_C|^* = \mathbb{P}T_0JC$ with center $|\omega_C \otimes \alpha|^* = \mathbb{P}T_0P$. Since $q_{\pi^*M} = \pi^*(q_M) = \pi^*(\cup_{\delta\in|M|}\langle\delta\rangle)$, we see that q_{π^*M} contains $\mathbb{P}T_0P \subset \mathbb{P}T_0J\widetilde{C}$ and the multiplicity of $P.\widetilde{\Theta}'_{\pi^*M^{-1}}$ at 0 is at least 3. Since this multiplicity is even, we have $P.\widetilde{\Theta}'_{\pi^*M^{-1}} \in |2\Xi|_{00}$. ☐

4. The base locus of $L|_{\widetilde{\Sigma}}$

In this section we prove Proposition 5 and Theorem 3. We will use the divisors of $|2\Xi|_{00}$ that we constructed in Section 3. We have

Proposition 4.1. *If $M \in Pic^{g-1}C$ is such that $h^0(C,M) = h^0(\widetilde{C},\pi^*M) = 2$, then*

$$\rho^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}}) = D_{\pi^*M} + D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} + 2\Delta'$$

where D_{π^*M} is defined as in Lemma 2.1. Furthermore, the divisors $D_{\pi^*M} - \Delta'$ and $D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} - \Delta'$ are effective with respective supports

$$\{(p,q) : h^0(C, M \otimes \mathcal{O}_C(-\pi p - \pi q)) > 0\}$$

and

$$\{(p,q) : h^0(C, \omega_C \otimes M^{-1} \otimes \mathcal{O}_C(-\pi p - \pi q)) > 0\}$$

Proof : First note that $P.\widetilde{\Theta}'_{\pi^*M^{-1}}$ does not contain Σ. Indeed, for general points p and q in $\widetilde{C}$, we have $h^0(\widetilde{C},\pi^*M \otimes \mathcal{O}_{\widetilde{C}}(p+q)) = 2$ and $h^0(\widetilde{C},\pi^*M \otimes \mathcal{O}_{\widetilde{C}}(-\sigma p - \sigma q)) = 0$. Therefore p and q are base points for $|\pi^*M \otimes \mathcal{O}_{\widetilde{C}}(p+q)|$ and $h^0(\widetilde{C},\pi^*M \otimes \mathcal{O}_{\widetilde{C}}(p+q-\sigma p-\sigma q)) = 0$.

Restricting to fibers of p_1 and p_2 and using the See-Saw Theorem, we see that $D_{\pi^*M} + D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} + 2\Delta' \in |\omega_{\widetilde{C}^2}(-2\Delta+2\Delta)|$. Hence, by Lemma 2.3 and Proposition 3.1, the divisor $D_{\pi^*M} + D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} + 2\Delta'$ is linearly equivalent to $\rho^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}})$. Let B and B' be the respective base loci of $|M|$ and $|\omega_C \otimes M^{-1}|$. By definition the support of $\rho^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}})$ is the set

$$\{(p,q) : h^0(\widetilde{C},\pi^*M \otimes \mathcal{O}_{\widetilde{C}}(p+q-\sigma p-\sigma q)) > 0\}$$

which, by Riemann-Roch and Serre Duality is equal to the set

$$\{(p,q) : h^0(\widetilde{C},\omega_{\widetilde{C}} \otimes \pi^*M^{-1} \otimes \mathcal{O}_{\widetilde{C}}(\sigma p+\sigma q-p-q)) > 0\}\ .$$

Therefore the support of $\rho^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}})$ contains the sets

$$\{(p,q) : h^0(C, \pi^*M \otimes \mathcal{O}_{\widetilde{C}}(-\pi^*B - p - q)) > 0\} = D_{\pi^*M\otimes\mathcal{O}_{\widetilde{C}}(-\pi^*B)},$$

$$\{(p,q) : h^0(C, \omega_{\widetilde{C}} \otimes \pi^*M^{-1} \otimes \mathcal{O}_{\widetilde{C}}(-\pi^*B' - \pi p - \pi q)) > 0\} = D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}\otimes\mathcal{O}_{\widetilde{C}}(-\pi^*B')}$$

and the support of $p_1^*\pi^*B + p_2^*\pi^*B + p_1^*\pi^*B' + p_2^*\pi^*B'$. Therefore, when B and B' are reduced, we have $\rho^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}}) = D_{\pi^*M\otimes\mathcal{O}_{\widetilde{C}}(-\pi^*B)} + D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}\otimes\mathcal{O}_{\widetilde{C}}(-\pi^*B')} + 2\Delta' + p_1^*\pi^*B + p_2^*\pi^*B + p_1^*\pi^*B' + p_2^*\pi^*B'$. When B or B' is not reduced, choose a (flat) one-parameter family of divisors $\{B_t + B'_t\}_{t\in T}$ whose general member is reduced and which has a special member $B_0 + B'_0$, $0 \in T$ with $B_0 = B$ and $B'_0 = B'$. Then the two families of divisors $\rho^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}\otimes\mathcal{O}_{\widetilde{C}}(\pi^*B - \pi^*B_t)})$ and $D_{\pi^*M\otimes\mathcal{O}_{\widetilde{C}}(-\pi^*B)} + D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}\otimes\mathcal{O}_{\widetilde{C}}(-\pi^*B')} + 2\Delta' + p_1^*\pi^*B_t + p_2^*\pi^*B_t + p_1^*\pi^*B'_t + p_2^*\pi^*B'_t$ define two divisors on $\widetilde{C}^2 \times T$ which are equal because they do not contain $\widetilde{C}^2 \times \{0\}$ and are equal outside it.

The other assertions of the Proposition are immediate. ☐

Proposition 4.2. *Let $M \in Pic^{g-1}C$ be such that $h^0(M) = h^0(\pi^*M) = 2$. Let $s \in \Gamma_{00}$ be such that $Z(s) = P.\widetilde{\Theta}'_{\pi^*M^{-1}}$. Then, with the notation of Lemma 2.3 we have $q(s) = \pi^*q_M$.*

Proof : We have, by Proposition 4.1, that

$$\rho^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}}) = D_{\pi^*M} + D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} + 2\Delta' \,.$$

Noting that $D_{\pi^*M} \cup D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} = \{(u,v) : \langle \pi u + \pi v\rangle \subset q_M\} \cup \Delta' = \{(u,v) : \langle u + v\rangle \subset \pi^*q_M\}$ as sets, it follows from Lemma 2.3 that $\pi^*q_M = q(s)$. ☐

Let

$$\tau_2 : \Gamma_0 \longrightarrow H^0(\mathbb{P}T_0P, \mathcal{O}_{\mathbb{P}T_0P}(2))$$

be the map which to $s \in \Gamma_0$ associates the quadric term of its Taylor expansion at 0. Then τ_2 is onto because, since (P, Ξ) is indecomposable, every quadric of rank 1 can be obtained as the tangent cone at the origin to a divisor $\Xi_\gamma + \Xi_{\gamma^{-1}}$ for some $\gamma \in P$. We have

Lemma 4.3. *For $s \in \Gamma_0$,*

$$s \in \Gamma_{00} \Longleftrightarrow q(s) \supset \mathbb{P}T_0P \,.$$

Proof: By Lemma 2.3 part 3 and with the notation there, if $s \in \Gamma_0 \backslash \Gamma_{00}$, then $\tau_{Z(s)} = q(s) \cap \mathbb{P}T_0P$. So the projectivizations of the two maps τ_2 and $s \mapsto (\rho^*s)|_{\mathbb{P}T_0P} \in I_2(\widetilde{C})|_{\mathbb{P}T_0P}$ are equal. Hence, there exists $\lambda \in \mathbb{C}^*$ such that, for every $s \in \Gamma_0$, we have $\lambda\tau_2(s) = (\rho^*s)|_{\mathbb{P}T_0P}$. So

$$s \in \Gamma_{00} \Leftrightarrow \tau_2(s) = 0 \Leftrightarrow (\rho^*s)|_{\mathbb{P}T_0P} = 0 \Leftrightarrow q(s) \supset \mathbb{P}T_0P \,.$$

☐

Let $I_2(\widetilde{C}, \alpha)$ be the subvector space of $I_2(\widetilde{C})$ consisting of elements which vanish on $\mathbb{P}T_0P$. By the above lemma and because all elements of Γ_0 are even, the map ρ^* sends Γ_{00} into the subspace $I_2(\widetilde{C}, \alpha)^+$ of σ-invariant elements of $I_2(\widetilde{C}, \alpha)$. We have

Lemma 4.4. *The subspace $I_2(\widetilde{C}, \alpha)^+$ is equal to $I_2(C) \overset{\pi^*}{\subset} I_2(\widetilde{C})$.*

Proof : The σ-invariant and σ-anti-invariant parts of $H^0(\widetilde{C},\omega_{\widetilde{C}})$ are, respectively, $H^0(C,\omega_C)$ and $H^0(C,\omega_C\otimes\alpha)$. Therefore, in the decomposition

$$S^2H^0(\widetilde{C},\omega_{\widetilde{C}})=S^2H^0(C,\omega_C)\oplus H^0(C,\omega_C)\otimes H^0(C,\omega_C\otimes\alpha)\oplus S^2H^0(C,\omega_C\otimes\alpha),$$

the space $S^2H^0(C,\omega_C)\oplus S^2H^0(C,\omega_C\otimes\alpha)$ is the σ-invariant part of $S^2H^0(\widetilde{C},\omega_{\widetilde{C}})$. So $S^2H^0(C,\omega_C)$ is the subspace of σ-invariant elements of $S^2H^0(\widetilde{C},\omega_{\widetilde{C}})$ which vanish on $\mathbb{P}T_0P$. Therefore $I_2(\widetilde{C},\alpha)^+$ is contained in $S^2H^0(C,\omega_C)$ and $I_2(\widetilde{C},\alpha)^+$ is the subspace of elements of $S^2H^0(C,\omega_C)$ which vanish on $\kappa\widetilde{C}$. This is precisely $I_2(C)$. $\square$

Let W^1_{g-1} be the subvariety of $Pic^{g-1}C$ parametrizing invertible sheaves with $h^0(C,M)\geq 2$. We have

Corollary 4.5. *The pull-back*

$$\rho^*:\Gamma_{00}\longrightarrow I_2(C)\left(\overset{\pi^*}{\subset} I_2(\widetilde{C})\right)$$

is onto.

Proof : Since all quadrics of rank four containing κC are of the form q_M for some $M\in W^1_{g-1}$ and a general such M has the properties required in Proposition 4.2, this follows from Proposition 4.2 and the fact that quadrics of rank four generate $|I_2(C)|$ (see [G] and [SV]). $\square$

Note that, since all elements of Γ_0 are even, the map ρ^* sends Γ_0 into the subspace $I_2(\widetilde{C})^+$ of σ-invariant elements of $I_2(\widetilde{C})$. We have

Corollary 4.6. *The pull-back*

$$\rho^*:\Gamma_0\longrightarrow I_2(\widetilde{C})^+\left(\subset I_2(\widetilde{C})\right)$$

is surjective.

Proof : Take the σ-invariants of the exact sequence

$$0\longrightarrow I_2(\widetilde{C})\longrightarrow S^2H^0(\widetilde{C},\omega_{\widetilde{C}})\longrightarrow H^0(\widetilde{C},\omega_{\widetilde{C}}^{\otimes 2})\longrightarrow 0$$

to obtain the exact sequence

$$0\longrightarrow I_2(\widetilde{C})^+\longrightarrow S^2H^0(C,\omega_C)\oplus S^2H^0(C,\omega_C\otimes\alpha)\longrightarrow H^0(C,\omega_C^{\otimes 2})\longrightarrow 0.$$

Taking the quotient of this sequence by $I_2(C)$, we obtain

$$0\longrightarrow \frac{I_2(\widetilde{C})^+}{I_2(C)}\longrightarrow H^0(C,\omega_C^{\otimes 2})\oplus S^2H^0(C,\omega_C\otimes\alpha)\longrightarrow H^0(C,\omega_C^{\otimes 2})\longrightarrow 0$$

or

$$\frac{I_2(\widetilde{C})^+}{I_2(C)}\xrightarrow{\cong} S^2H^0(C,\omega_C\otimes\alpha)$$

where the isomorphism is obtained from the restriction to $\mathbb{P}T_0P$. By Corollary 4.5, we have the equality $\rho^*\Gamma_{00}=I_2(C)$ and hence the embedding $\frac{\rho^*\Gamma_0}{I_2(C)}\hookrightarrow\frac{I_2(\widetilde{C})^+}{I_2(C)}=S^2H^0(C,\omega_C\otimes\alpha)$ which is equal to the map $s\mapsto\rho^*(s)|_{\mathbb{P}T_0P}$. The projectivization of this map is equal to the projectivization of τ_2 (see Lemmas 2.3 and 4.3). Since τ_2 is surjective, we have $\frac{\rho^*\Gamma_0}{I_2(C)}=\frac{I_2(\widetilde{C})^+}{I_2(C)}$ and $\rho^*\Gamma_0=I_2(\widetilde{C})^+$. $\square$

Remark 4.7. **(C. Pauly)** *Counting dimensions, one easily deduces from the above two corollaries that* Γ'_{00} *is in fact the subspace of elements of* Γ *(and not just* Γ_{00}*) vanishing on* Σ.

The following implies Theorem 3

Proposition 4.8. *The inverse image by* $\widetilde{\rho}$ *of the support of the base locus of* $L|_{\widetilde{\Sigma}}$ *is the set of elements* (p,q) *of* $\widetilde{C}^2$ *such that* $\langle \pi p + \pi q\rangle$ *is contained in the intersection of the quadrics containing the canonical curve* κC*. In particular, if* C *is not trigonal, then the base locus of* L *does not intersect* $\widetilde{\Sigma}$.

Proof : By Proposition 3.2 and Corollary 4.5, the base locus of $L|_{\widetilde{\Sigma}}$ is supported on $\widetilde{\Sigma} \cap \left(\cap_{M\in W^1_{g-1}} \epsilon_*^{-1}(P.\widetilde{\Theta}'_{\pi^*M^{-1}})\right)$ where $\epsilon_*^{-1}(P.\widetilde{\Theta}'_{\pi^*M^{-1}}) = \epsilon^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}}) - 4\mathcal{E}$. We have $\widetilde{\rho}^*(\epsilon_*^{-1}(P.\widetilde{\Theta}'_{\pi^*M^{-1}})) = \rho^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}}) - 4\Delta'$. Since $\langle \pi p + \pi q\rangle \subset q_M$ is equivalent to $h^0(C, M\otimes \mathcal{O}_C(-\pi p - \pi q)) > 0$ or $h^0(C, \omega\otimes M^{-1}\otimes \mathcal{O}_C(-\pi p - \pi q)) > 0$, it follows from Proposition 4.1 that the inverse image $\widetilde{\rho}^*\left(\cap_{M\in W^1_{g-1}} \epsilon_*^{-1}(P.\widetilde{\Theta}'_{\pi^*M^{-1}})\right)$ is supported on the set of elements (p,q) of $\widetilde{C}^2$ such that $\langle \pi p + \pi q\rangle$ is contained in q_M for all $M \in W^1_{g-1}$. Since the quadrics of the form q_M generate $|I_2(C)|$ (see [G] and [SV]) and the base locus of $|I_2(C)|$ in the canonical space $|\omega_C|^*$ does not contain any secants to κC for C non-trigonal (see [ACGH] page 124), the proposition follows. ☐

Corollary 4.9. *The dimension of* Γ'_{00} *is equal to* $2^p - 2 - p^2 + p$*. The codimension of* $\mathcal{Q}'_{00}$ *in* $\mathcal{Q}_{00}$ *is at least* $g-3$.

Proof : By Lemma 4.4, we have $\Gamma'_{00} = Ker(\rho^* : \Gamma_{00} \longrightarrow I_2(C) \overset{\pi^*}{\subset} I_2(\widetilde{C}))$. Since ρ^* maps Γ_{00} onto $I_2(C)$ by Corollary 4.5, the dimension of Γ'_{00} is equal to $\dim(\Gamma_{00}) - \dim(I_2(C)) = 2^p - 1 - \frac{1}{2}p(p+1) - \frac{1}{2}(g-2)(g-3) = 2^p - 1 - \frac{1}{2}p(p+1) - \frac{1}{2}(p-1)(p-2) = 2^p - 2 - p^2 + p$.

The linear system $\widetilde{\rho}^*L$ contains the divisors $\widetilde{\rho}^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}}) - 4\Delta'$ for $M \in W^1_{g-1}$. From now on we suppose that $h^0(C,M) = h^0(\widetilde{C}, \pi^*M) = 2$. By Proposition 4.1, we have $\widetilde{\rho}^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}}) - 4\Delta' = D_{\pi^*M} - \Delta' + D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} - \Delta'$. Therefore the restriction of $\widetilde{\rho}^*L$ to $\Delta' \cong \widetilde{C}$ contains the divisors $(D_{\pi^*M} - \Delta')|_{\Delta'} + (D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} - \Delta')|_{\Delta'}$. We will prove that the map $M \mapsto (\widetilde{\rho}^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}}) - 4\Delta')|_{\Delta'}$ is generically finite. This will imply that the set of such divisors on $\Delta' \cong \widetilde{C}$ has dimension $g-4$ and the codimension of $\mathcal{Q}'_{00}$ in $\mathcal{Q}_{00}$ is at least $g-4+1 = g-3$. We let R and R' be the ramification divisors of the natural maps $C \longrightarrow |M|^* \cong \mathbb{P}^1$ and $C \longrightarrow |\omega_C \otimes M^{-1}|^* \cong \mathbb{P}^1$ respectively

First suppose C non-trigonal. If C is bielliptic and $g \geq 6$, we choose M in the component of W^1_{g-1} whose general elements are base-point-free. For M general, the linear systems $|M|$ and $|\omega_C \otimes M^{-1}|$ have no base points and $(D_{\pi^*M} - \Delta')|_{\Delta'} = \pi^*R$, $(D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} - \Delta')|_{\Delta'} = \pi^*R'$. It follows that the map $M \mapsto (\widetilde{\rho}^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}}) - 4\Delta')|_{\Delta'}$ is generically finite.

Now suppose C trigonal. Since $g \geq 5$, the curve C has a unique linear system of degree 3 and dimension 1 and we denote the associated invertible sheaf of degree 3 by M_0. Choose a general effective divisor N of degree $g-4$ on C and put $M = M_0 \otimes \mathcal{O}_C(N)$. We have $\widetilde{\rho}^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}}) - 4\Delta' = D_{\pi^*M_0} - \Delta' + p_1^*(\pi^*N) + p_2^*(\pi^*N) + D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} - \Delta'$ and $(\widetilde{\rho}^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}}) - 4\Delta')|_{\Delta'} = (D_{\pi^*M_0} - \Delta')|_{\Delta'} + 2\pi^*N + (D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} - \Delta')|_{\Delta'}$ (where we identify Δ' with $\widetilde{C}$). Since N is general, the linear system $|\omega_C \otimes M^{-1}|$ has no base points and $(D_{\omega_{\widetilde{C}}\otimes\pi^*M^{-1}} - \Delta')|_{\Delta'} = \pi^*R'$. It follows once more that the map $M \mapsto (\widetilde{\rho}^*(P.\widetilde{\Theta}'_{\pi^*M^{-1}}) - 4\Delta')|_{\Delta'}$ is generically finite. ☐

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