

# The Faber-Krahn inequality

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In this note we prove the following classical eigenvalue inequality, due separately to Faber [F] and Krahn [K].

**Theorem 1.** *Let  $D \subset \mathbb{R}^n$  be a bounded domain and let  $B$  be the ball centered at the origin with  $\text{Vol}(D) = \text{Vol}(B)$ . Then  $\lambda_1(D) \geq \lambda_1(B)$ , with equality if and only if  $D = B$  almost everywhere.*

Here  $\lambda_1(D)$  is the first eigenvalue of the Laplacian, with Dirichlet boundary conditions. The proof below borrows much from the proof in Chavel's book [Ch], but is a little less general and simplifies some of the notation.

*Proof.* Recall variational characterization of the first eigenvalue:

$$\lambda_1(D) = \inf \left\{ \frac{\int_D |\nabla u|^2 dV}{\int_D u^2 dV} \mid u \in C_0^2(D) \right\}. \quad (1)$$

By the Courant nodal domain theorem, we can take a test function for the Rayleigh quotient be non-negative. Let  $u$  be a test function, and for  $0 \leq t \leq \hat{u} = \max(u)$  let  $D_t = \{u > t\}$ .

Now we define a comparison function  $u_* : B \rightarrow [0, \infty)$  as follows. First let  $B_t$  be the ball centered at the origin with  $\text{Vol}(B_t) = \text{Vol}(D_t)$ . Then let  $u_*$  be the radially symmetric function such that  $B_t = \{u_* > t\}$ . By the co-area formula,

$$\int_t^{\hat{u}} \int_{\partial D_\tau} \frac{dA}{|\nabla u|} d\tau = \text{Vol}(D_t) = \text{Vol}(B_t) = \int_t^{\hat{u}} \int_{\partial B_\tau} \frac{dA}{|\nabla u_*|} d\tau.$$

Differentiating with respect to  $t$  gives us

$$\int_{\partial D_t} \frac{dA}{|\nabla u|} = \int_{\partial B_t} \frac{dA}{|\nabla u_*|} \quad (2)$$

for all  $t$ . Then

$$\begin{aligned} \int_D u^2 dV &= \int_0^{\hat{u}} \int_{\partial D_t} \frac{u^2 dA}{|\nabla u|} dt = \int_0^{\hat{u}} t^2 \int_{\partial D_t} \frac{dA}{|\nabla u|} dt \\ &= \int_0^{\hat{u}} t^2 \int_{\partial B_t} \frac{dA}{|\nabla u_*|} dt = \int_B u_*^2 dV. \end{aligned} \quad (3)$$

Now, for  $0 \leq t \leq \hat{u}$  let

$$\psi(t) = \int_{D_t} |\nabla u|^2 dV, \quad \psi_*(t) = \int_{B_t} |\nabla u_*|^2 dV.$$

By the co-area formula

$$\psi' = - \int_{\partial D_t} |\nabla u| dA, \quad \psi_*' = - \int_{\partial B_t} |\nabla u_*| dA.$$

We use the Cauchy-Schwarz inequality, the isoperimetric inequality, and the fact that the normal derivative of  $u_*$  is constant on  $\partial B_t$  to see

$$\begin{aligned} \left( \int_{\partial D_t} |\nabla u| dA \right) \left( \int_{\partial D_t} \frac{dA}{|\nabla u_*|} \right) &\geq \left( \int_{\partial D_t} dA \right)^2 = (\text{Area}(\partial D_t))^2 \\ &\geq (\text{Area}(\partial B_t))^2 = \left( \int_{\partial B_t} |\nabla u_*| dA \right) \left( \int_{\partial B_t} \frac{dA}{|\nabla u_*|} \right). \end{aligned}$$

We use equation (2) to cancel the common factor of

$$\int_{\partial D_t} \frac{dA}{|\nabla u|} = \int_{\partial B_t} \frac{dA}{|\nabla u_*|},$$

and so

$$-\psi' = \int_{\partial D_t} |\nabla u| dA \geq \int_{\partial B_t} |\nabla u_*| dA = -\psi'_*.$$

Integrating this last differential inequality and using  $\psi(\hat{u}) = 0 = \psi_*(\hat{u})$  we see

$$\int_D |\nabla u|^2 dV = \psi(0) \geq \psi_*(0) = \int_B |\nabla u_*|^2 dV.$$

Combine this inequality with (3) and (1) to give the desired inequality on the eigenvalues:

$$\lambda_1(D) \geq \lambda_1(B).$$

Moreover, equality of the eigenvalues forces the level sets  $\partial D_t$  to all be spheres centered at the origin. Also, the equality case of the Cauchy-Schwarz inequality forces  $|\nabla u|$  to be constant on the level set  $\partial D_t$ . Thus  $u$  must be radially symmetric, and so in this case  $u = u_*$ .  $\square$

Observe that the proof really only requires that the isoperimetric domains in the ambient space are geodesic balls, and so the same proof applies to domains in hemispheres and hyperbolic space. In fact, the proper way to state the Faber-Krahn inequality is to say that a geometric isoperimetric inequality:

$$\text{Vol}(D) = \text{Vol}(B) \Rightarrow \text{Area}(\partial D) \geq \text{Area}(\partial B),$$

where  $B$  is a ball, implies a physical isoperimetric inequality:

$$\text{Vol}(D) = \text{Vol}(B) \Rightarrow \lambda_1(D) \geq \lambda_1(B).$$

This is actually the statement Chavel gives in his book [Ch].

Also, a byproduct of the proof is the fact that the first Dirichlet eigenfunction of the ball is radially symmetric. This allows one to explicitly compute the first eigenfunction (and its eigenvalue) in terms of Bessel functions.

## References

- [Ch] I. Chavel. *Eigenvalues in Riemannian Geometry*. Series in Pure and Applied Mathematics. 115. Academic Press, Inc., Orlando, 1984.
- [F] C. Faber. *Beweiss, dass unter allen homogenen Membrane von gleicher Fläche und gleicher Spannung die kreisförmige die tiefsten Grundton gibt*. Sitzungsber.-Bayer. Akad. Wiss., Math.-Phys. Munich. (1923), 169–172.
- [K] E. Krahn. *Über eine von Rayleigh formulierte Minmaleigenschaft des Kreises*. Math. Ann. **94** (1925), 97–100.