Research Interests: Jesse Ratzkin

1 Introduction

My research lies at the intersection of geometry and analysis, concentrating on variational geometry. Geometric variational problems study manifold structures which are extrema, or critical points, of a geometrically defined functional. These problems include:

- The Dirichlet energy is the integral of the square norm of the gradient of a function. Critical points give us eigenvalues and eigenfunctions of the Laplace operator.
- Given a topological surface Σ, one can look for minima, or critical points, of the area functional, evaluated on embeddings of Σ into a manifold of dimension $n \geq 3$. Critical points of this functional are called minimal surfaces, and they satisfy a well-studied, quasi–linear, elliptic partial differential equation (PDE). More generally, one can look at critical points of the k–dimensional volume of k–dimensional submanifold in an n–dimensional manifold. These critical points are minimal submanifolds, and also satisfy a quasi–linear, elliptic PDE.
- In the case of $(n-1)$–dimensional manifolds in n–dimensional manifolds (hypersurfaces), one can modify the previous example of minimal submanifolds by adding a volume constraint. More specifically, one fixes the n–dimensional volume the submanifold encloses. These hypersurfaces satisfy a different, but closely related elliptic PDE, and are called constant mean curvature (CMC) hypersurfaces.
- One can study the integral of the scalar curvature of a metric, normalized by the volume to the appropriate power, restricted to a given conformal class. Critical points of the functional are constant scalar curvature metrics. The conformal factor of such a metric satisfies a well–studied, semi–linear, elliptic PDE.

These topics, though they appear diverse, relate to each other in surprising ways. For instance, one can often apply the same proof in both the mean curvature and the scalar curvature setting (compare [KMP] and [MPU], or [CGS] and [KKS]).

One can place a problem in each of the topics listed above into one of several categories: constructing examples (existence), studying deformations and moduli (uniqueness), and qualitative/quantitative statements about solutions (estimates). One should not view existence, uniqueness, and estimates in isolation; instead they are deeply related. For instance, one often needs to obtain sharp (or close to sharp) estimates in order to construct solutions [S2, MPa]. Constructing new examples will often reveal new and interesting features of the appropriate moduli space [MPP, R1]. Finally, general properties of the appropriate moduli space should inform one's construction of examples [MPa, MPP, R1, R2].

In the sections below I will outline my previous work and ideas for new projects in the study of eigenvalues, CMC surfaces, scalar curvature, and related curvature problems.

2 Eigenvalues

Given a compact domain $\Omega \subset M$, where (M, g) is a Riemannian manifold, the first Dirichlet eigenvalue is defined as the infimum of the Rayleigh quotient:

$$\lambda_1(\Omega) := \inf \left\{ \frac{\int_\Omega |du|^2}{\int_\Omega u^2} \right\},$$

where u is a smooth function whose support is compactly contained in Ω. The literature contains many estimates, upper and lower, for $\lambda_1(\Omega)$, depending on the geometry of Ω, see [C] for an overview. One particular lower estimate for $\lambda_1(\Omega)$ is the Faber–Krahn inequality: if Ω is contained in a space form and $\mathrm{Vol}(\Omega) = \mathrm{Vol}(B_r)$, where B_r is a ball of radius r, then $\lambda_1(\Omega) \geq \lambda_1(B_r)$, with equality if and only if Ω is a ball.

2.1 Previous work

In recent work, A. Treibergs and I [RT2] proved a Faber–Krahn type inequality for domains in a spherical wedge. This result extends a theorem of Payne and Weinberger [PW] in the case of domains contained in a planar wedge. More precisely, let (ρ, θ) be polar coordinates on S^2, centered at the north pole, let $\mathcal{W} = \{(\rho, \theta) \mid 0 \leq \theta \leq \pi/\alpha\}$ for some $\alpha \geq 1$, and let Ω be a compact domain in $\mathcal{W}$. Observe that

$$w(\rho, \theta) := \tan^\alpha(\rho/2) \sin(\alpha\theta)$$

is a positive harmonic function on $\mathcal{W}$ with zero boundary data. Let $S(r^*) = \{(\rho, \theta) \in \mathcal{W} \mid 0 \leq \rho \leq r^*\}$, choosing r^* so that

$$\int_\Omega w^2 = \int_{S(r^*)} w^2.$$

Then we prove $\lambda_1(\Omega) \geq \lambda_1(S(r^*))$, with equality if and only if $\Omega = S(r^*)$. Our proof is based on a isoperimetric–type inequality. As an application we improve our proof in [RT1] regarding Brownian motion.

2.2 Future projects

It is interesting that our proof does not extend to wedge domains contained in the hyperbolic plane. One can define a similar positive harmonic function: $w(\rho, \theta) = \tanh^\alpha(\rho/2) \sin(\alpha\theta)$. However, the estimates we use to prove our isoperimetric inequality do not hold, and it seems unlikely that one can prove the relevant estimates using the harmonic weight $\tanh^\alpha(\rho/2) \sin(\alpha\theta)$. Instead, in the hyperbolic case one must pick a different harmonic weight function, which we haven't identified yet.

Additionally, our techniques in [RT2] are particular to 2 dimensions. Are there similar eigenvalue estimates for higher dimensional wedge domains? One can examine domains contained in a cone over a equatorial slice (see [RT1] for a precise definition of these spherical cones), and apply our coning construction to the sharp estimate of [RT2]. However, it is not yet clear precisely when we obtain interesting eigenvalue bounds with this technique.

3 Constant mean curvature surfaces

Constant mean curvature (CMC) surfaces are critical points of the area functional, subject to volume constraint, a variational formulation which leads to an elliptic, quasilinear PDE for the surface. I study CMC surfaces with finite topology which are noncompact and properly embedded (or, more generally, almost embedded, see [KKS] for a definition) in $\mathbb{R}^3$. Any end of such a surface has a well–defined asymptote [KKS].

3.1 Previous work

My work in this area started with the gluing construction of my thesis [R1]. In this construction, one truncates two ends of two CMC surfaces, where the truncated ends have the same asymptotes, and glues the truncated surfaces together near the truncations. If the original surfaces have k_1 and k_2 ends, respectively, the new surface one constructs has $k_1 + k_2 - 2$ ends. Additionally, the angle parameter which rotates one surface about the asymptotic axis along which one is gluing, while holding the other surface fixed, remains free during the construction. Varying this angle parameter yields a Dehn twist in the conformal class of the new, glued surface, giving us a non-contracible loop in the moduli space of CMC surfaces with genus g and k ends, $k \geq 4$.

My further work concentrates on a condition called *nondegeneracy*. A CMC surface $f : \Sigma \to \mathbb{R}^3$ is called nondegenerate if the only L^2 solution to the linearized mean curvature equation $\mathcal{L}(u) := (\Delta + |A_f|^2)(u) = 0$ is the zero function. Here Δ is the Laplace–Beltrami operator and $|A_f|$ is the norm of the second fundamental form. In general, one should think of nondegeneracy as an infinitesimal rigidity condition; it implies local rigidity in that (by [KMP]) if $f(\Sigma)$ is nondegenerate then there are no other CMC surfaces near $f(\Sigma)$, in the appropriate topology, with the same asymptotes. In joint work with Große–Brauckmann, Korevaar, Kusner, and Sullivan [GKKRS, KKR] we have proved genus zero CMC surfaces which are contained in a solid slab are nondegenerate. By the moving planes technique [KKS] the solid slab condition forces the surface to have a plane of reflection symmetry, and that each half of the surface one one side of the symmetry plane is a graph. Such a surface is called coplanar. Some applications of our nondegeneracy theorem include regularity of the moduli space (along the lines of [KMP]) and the showing that the classifying map of [GKS2] is a diffeomorphism.

3.2 Future projects

Our work regarding nondegeneracy leads to many natural open questions. For instance, our results are the only general theorems showing that all members of a wide class of CMC surfaces are nondegenerate. Is nondegeneracy a truly generic condition? Can one bound the size (in terms of, for instance, Hausdorff dimension) of the set of degenerate surfaces? For that matter, are there any degenerate CMC surfaces? To date, there are no examples. Also, can one extend the results of [GKKRS] to CMC surfaces with other (*e.g.* tetrahedral) symmetries?

The moduli space $\mathcal{M}_{g,k}$ of CMC surfaces with genus g and k ends (see [KMP] for a precise definition) is still poorly understood. For instance, there is a natural forgetful map from $\mathcal{M}_{g,k}$ to the space of conformal structures, which only remembers the conformal structure of the surface [Kus, MPP]. This forgetful map is onto in the genus zero case, but not in general. Which conformal structures are realized as CMC surfaces for higher genus? One can construct representatives of many conformal classes by gluing, but all these conformal classes are close to singular, with either small necks or long interior annuli. It might be possible to connect some of these near–singular conformal classes using a continuity arguments. In the genus zero case, it is reasonable to believe that the fiber (*i.e.* preimage of a point) of the forgetful map is finite. If the fiber is finite, R. Kusner (motivated by the end to end gluing of [R1]) has suggested a relation between its cardinality and the number of pants decompositions of the surface.

4 Scalar curvature and other related curvature problems

Let (M, g_0) be an n–dimensional Riemannian manifold, and denote scalar curvature by $R = R(g)$. The metrics g which are conformal to g_0, written $g = u^{4/(n-2)} g_0$, having constant scalar curvature are critical points of the functional

$$Q(u) = \frac{\int R(g) dV_g}{(\text{Vol}(M, g))^{(n-2)/n}} = \frac{\int \left(\frac{4(n-1)}{n-2}\right) |\nabla u|^2 + R(g_0) u^2 dV_{g_0}}{\left(\int u^{2n/(n-2)} dV_{g_0}\right)^{(n-2)/n}}.$$

Her u is a smooth, positive function. By [S1], if M is compact then any conformal class $[g_0]$ contains a constant scalar curvature metric. Conformal factors u associated to constant scalar curvature metrics satisfy the semi-linear, elliptic PDE

$$\Delta u + \frac{n-2}{4(n-1)} R(g_0) u = \lambda u^{(n+2)/(n-2)},$$

where λ is a constant and Δ is the Laplace–Beltrami operator. This PDE has critical growth on the right hand side. As is common in PDEs with critical growth, a sequence tending towards a solution can blow up along a closed set Λ, leading to the singular Yamabe problem (see for instance [AM, SY, MPU]): given a compact Riemannian manifold $(\bar{M}, g_0)$ and a closed set $\Lambda \subset \bar{M}$, find a conformal metric $g = u^{4/(n-2)}$ with constant scalar curvature which is complete on $M = \bar{M} \backslash \Lambda$.

4.1 Previous work

In the case that $\Lambda = \{p_1, \ldots, p_k\}$ is a finite set of points, the singular Yamabe metrics behave much like the constant mean curvature surfaces discussed in the previous section. In particular, the metrics have a definite asymptotic structure approaching point p_j of the singular set [CGS]. As in the mean curvature case, one can truncate two ends of such a metric with matching asymptotics, and gluing the resulting truncated metrics together, producing a new constant scalar curvature metric [R2]. If the original metrics g_1 and g_2 had k_1 and k_2 singular points, respectively, then the new metrics has $k_1 + k_2 - 2$ singular points.

Additionally, R. Kusner and I have a continuing joint project in which we try to bound the asymptotics of a singular Yamabe metric on a finitely punctured sphere, in the presence of symmetry.

4.2 Future projects

Schoen and Yau [SY] showed that if (M, g) is locally conformally flat, singular Yamabe metric with positive scalar curvature, then $M = \bar{M} \backslash \Lambda$ and the Hausdorff dimension of Λ is at most

$(n-2)/2$. Schoen [S2] constructs many singular Yamabe metrics on $S^n\backslash\Lambda$, where Λ is a finite set of points. His construction also includes more general singular sets, some of which have a structure like the Cantor set. (One can also see [MPa] for other singular sets which are unions of submanifolds.) However, it is not known how general the singular set can be. For instance, can the singular set Λ be any closed set of Hausdorff dimension at most $(n-2)/2$? Or are there other restrictions on Λ?

Additionally, there are fully–nonlinear analogs of scalar curvature, which are called k–curvature [V, CGY, GV]. The k–curvature σ_k of (M, g) is the kth elementary symmetric function evaluated on the eigenvalues of the Schouten tensor

$$A_{ij} = \frac{1}{n-2}\left(\mathrm{Ric}_{ij} - \frac{R}{2(n-1)} g_{ij}\right),$$

where Ric is the Ricci tensor. The Schouten tensor is a fundamental part of the full Riemann curvature tensor Riem, encoding the parts of Riem which are not pointwise conformally invariant. Observe that σ_1 is a multiple of R, and so σ_k provides a fully nonlinear natural generalization of R for $k > 1$. The hope is that, for $k > 1$, the integral of σ_k leads to new and interesting conformal invariants, as it does for $k = 2$ in dimension 4 [GV]. However, one difficulty is that, as in the case of scalar curvature, a sequence tending towards a constant k–curvature metric can blow up, leading to singularities. The possible singularities for k–curvature are not as well–understood as the case of scalar curvature. Some results should carry over, such as the k–curvature analog of Schoen's gluing construction [S2], at least if k is small enough. Conversely, some results seem much more difficult, such as the analog of the asymptotics theorem [CGS] (compare with the results of González [Gon1, Gon2]).

References

[AM] P. Aviles and R. McOwen. *Complete conformal metrics with constant negative scalar curvature in compact Riemannian manifolds.* Duke Math. J. 56:395–398, 1988.

[CGS] L. Caffarelli, B. Gidas and J. Spruck. *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth.* Comm. Pure Appl. Math. 42:271–297, 1988.

[CGY] S.-Y. Chang, M. Gursky and P. Yang. *An equation of Monge–Ampére type in conformal geometry, and four–manifolds of positive Ricci cuvature.* Ann. Math. 155:709–787, 2002.

[C] I. Chavel. *Eigenvalues in Riemannian Geometry.* Academic Press, 1984.

[Gon1] González, M. *Singular sets of a class of locally conformally flat manifolds.* Duke Math. J. 3:551–572, 2005.

[Gon2] González, M. *Classification of singularities for a subcritical fully nonlinear problem.* Pacific J. Math. 226:83–102, 2006.

[GKKRS] K. Große-Brauckmann, N. Korevaar, R. Kusner, J. Ratzkin, and J. Sullivan. *Coplanar constant mean curvature surfaces of genus zero are nondegenerate.* preprint.

[GKS1] K. Große-Brauckmann, R. Kusner and J. Sullivan. *Triunduloids: Embedded constant mean curvature surfaces with three ends and genus zero.* J. Reine Angew. Math. 564:35–61, 2003.

[GKS2] K. Große-Brauckmann, R. Kusner and J. Sullivan. *Coplanar constant mean curvature surfaces.* preprint. arXiv:math.DG/0509210.

[GV] M. Gursky and J. Viaclovsky. *Volume comparison and the σ_k–Yamabe problem.* Adv. Math. 187:447–487, 2004.

[KKR] N. Korevaar, R. Kusner and J. Ratzkin. *On the nondegeneracy of constant mean curvature surfaces.* Geom. Funct. Anal. 16:891–923, 2006.

[KKS] N. Korevaar, R. Kusner and B. Solomon. *The structure of complete, embedded surfaces with constant mean curvature.* J. Differential Geom. 30:465–503, 1989.

[Kus] R. Kusner. *Conformal structure and necksizes of embedded constant mean curvature surfaces* in Global theory of minimal surfaces, Clay Math. Proc. 2:585–596, Amer. Math. Soc. 2005.

[KMP] R. Kusner, R. Mazzeo and D. Pollack. *The moduli space of complete embedded constant mean curvature surfaces.* Geom. Funct. Anal. 6:120–137, 1996.

[MPa] R. Mazzeo and F. Pacard. *Constant scalar curvature metrics with isolated singularities.* Duke Math. J. 99:353–418, 1999.

[MPP] R. Mazzeo, F. Pacard and D. Pollack. *The conformal theory of Alexandrov embedded constant mean curvature surfaces in* $\mathbb{R}^3$. in Global theory of minimal surfaces, Clay Math. Proc. 2:525–559, Amer. Math. Soc. 2005.

[MPU] R. Mazzeo, D. Pollack and K. Uhlenbeck. *Moduli spaces for singular Yamabe metrics.* J. Amer. Math. Soc. 9:303–344, 1996.

[PW] L. Payne and H. Weinberger. *A Faber–Krahn inequality for wedge–like membranes.* J. Math. Phys. 39:182–188, 1960.

[R1] J. Ratzkin. *An end to end gluing construction for surfaces of constant mean curvature.* Ph.D. Thesis, University of Washington, 2001.

[R2] J. Ratzkin. *An end to end gluing construction for metrics of constant positive scalar curvature.* Indiana Univ. Math. J. 52:703–726, 2003.

[RT1] J. Ratzkin and A. Treibergs. *A capture problem in Brownian motion and eigenvalues of spherical domains.* to appear, Tran. Amer. Math. Soc.

[RT2] J. Ratzkin and A. Treibergs. *A Payne–Weinberger estimate for wedge domains on spheres.* preprint.

[S1] R. Schoen. *Conformal deformation of a Riemannian metric to constant scalar curvature.* J. Differential Geom. 20:479–495. 1984.

[S2] R. Schoen. *The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation.* Comm. Pure Appl. Math. 41:317–392, 1988.

[SY] R. Schoen and S. T. Yau. *Conformally flat manifolds, Kleinian groups and scalar curvature.* Invent. Math. 92:47–71, 1988.

[V] J. Viaclovsky. *Conformal geometry, contact geometry, and the calculus of variations.* Duke Math. J. 101:283–316, 2000.