A Finite Dimensional Approximation to a TV-L_p model for Image Denoising

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TV-Model

the TV- L^2 model(Rudin-Osher-Fatemi, 1992):

$$\min_{u \in \mathrm{BV}(\Omega)} |u|_{BV} + \frac{1}{2\lambda} \int_{\Omega} |u - f|^2 dx. \tag{1}$$

the TV- L^1 mode(Chan-Esedoglu, 2005; Nikolova, 2004):

$$\min_{u \in \mathrm{BV}(\Omega)} |u|_{BV} + \frac{1}{\lambda} \int_{\Omega} |u - f| dx. \tag{2}$$

Exact solution and its regularity are studied by William Allard(2008).

- TV- L^1 and TV- L^2 recovery.

Figure: original image

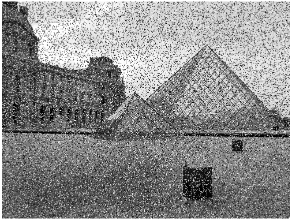

Figure: image + salt and pepper noise

L^1 is better sometimes

- TV- L^1 and TV- L^2 recovery.

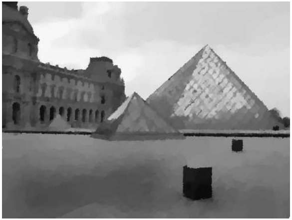

Figure: L^1 recovery

Figure: L^2 recovery

The TV-Lp Model

$$\min_{u\in \mathrm{BV}(\Omega)} \int_\Omega \sqrt{\epsilon + |\nabla u|^2} dx + \frac{1}{p\lambda}\int_\Omega (\eta + |u-f|^2)^{p/2} dx. \tag{3}$$

$$= \min_{u\in \mathrm{BV}(\Omega)} J(u) + \frac{1}{p\lambda}\int_\Omega (\eta + |u-f|^2)^{p/2} dx.$$

the following minimization:

$$\min_{u \in BV(\Omega)} \{|u|_{BV}, \quad \text{subject to } var(u-f) \leq \sigma_0^2\} \tag{4}$$

where $f = u_0 + \xi$ is a given noised image and $var(u-f)$ stands for the variance of $u-f$.

Motivation

In the discrete setting, suppose that $f = \{f_i, \cdots, f_n\}$ is a given image with $f_i = u_i + \xi_i$, where u_i are pixel values ,ξ_i are noise values. Suppose that ξ_i are IID, zero mean, variance σ^2.

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\xi_i - \bar{\xi})^2,$$

$E(s^2) = \sigma^2$. Note that $\bar{\xi} \approx 0$. Then(since $u - f = \xi$)

$$s^2 \simeq \frac{1}{n-1}\sum_{i=1}^{n}(\xi_i)^2 = \frac{1}{n-1}\sum_{i=1}^{n}(u_i - f_i)^2 \simeq \frac{1}{A_\Omega}\int_\Omega |u-f|^2 dx$$

where A_Ω stands for the area of Ω.

Biased estimator

Suppose that we have an *a prior* knowledge of the distribution of ξ_i. For example, ξ_i are independent identically distributed random variables with the same probability density function:

$$g_p(x) = \frac{C_p}{b} \exp\left(-\left|\frac{x}{b}\right|^p\right), p \geq 1 \tag{5}$$

with $C_p = \dfrac{p}{2\Gamma(1/p)}$ and b being an fixed parameter. It is easy to see that the mean $E(\xi_i) = 0$ and the variance

$$\sigma^2 = var(\xi_i) = \frac{\Gamma(3/p)}{\Gamma(1/p)} b^2. \tag{6}$$

Maximum Likelihood

For the given image $f = \{f_1, \cdots, f_n\}$, with the random variables $\xi_i, i = 1, \cdots, n$ which are iid with the same probability density function g_p, the event $\xi_i, i = 1, \cdots, n$ most likely happens when the joint probability

$$L(\xi_1, \xi_2, \cdots, \xi_{n-1}, \xi_n | \, b) = \left(\frac{C_p}{b} \right)^n \exp\left(- \frac{\sum_i^n |\xi_i|^p}{b^p} \right)$$

is maximized.

$$\hat{b} = \left(\frac{p}{n} \sum_{i=1}^{n} |\xi_i|^p \right)^{1/p}$$

Maximum Likelihood

$$\begin{aligned} var(u-f) &= \sigma^2 = \frac{\Gamma(3/p)}{\Gamma(1/p)} b^2 \simeq \frac{\Gamma(3/p)}{\Gamma(1/p)} \left(\frac{p}{n} \sum_i^n |\xi_i|^p \right)^{2/p} \\ &= \frac{\Gamma(3/p)}{\Gamma(1/p)} \left(\frac{p}{n} \sum_i^n |u_i - f_i|^p \right)^{2/p} \\ &\approx \frac{\Gamma(3/p)}{\Gamma(1/p)} \left(\frac{p}{A_\Omega} \int_\Omega |u-f|^p \right)^{2/p} \end{aligned}$$

New Model

$$\min_{u\in BV(\Omega)}\{|u|_{BV}, \quad \text{subject to } \frac{\Gamma(3/p)}{\Gamma(1/p)}\left(\frac{p}{A_\Omega}\int_\Omega |u-f|^p\right)^{2/p} \leq \sigma_0^2\} \tag{7}$$

Similar to the proof of the Chambolle-Lions result, the minimization is equivalent to the following

$$\min_{u\in BV(\Omega)} |u|_{BV} + \frac{1}{p\lambda}\int_\Omega |u-f|^p \tag{8}$$

for some λ dependent on σ_0 if σ_0 is not too big.

- What is the difference of the spline minimizer and the continuous minimizer in L^2?
- We assume f is bounded.

A Regularity result

Theorem

Fix $p \geq 1$ $\eta > 0$ *and* $\epsilon > 0$. *Suppose* f *is bounded and* u_f *is the minimizer of (3). Suppose that a spline space* $\mathcal{S}$ *contains* $S_d^r(\triangle)$ *for a degree* $d \geq 3r + 2$ *as a subspace. Let* S_f *be the minimizer of*

$$\min_{u \in \mathcal{S}} \int_\Omega \sqrt{\epsilon + |\nabla u|^2} dx + \frac{1}{p\lambda} \int_\Omega |u - f|^p dx \tag{9}$$

in $\mathcal{S}$. *Then*

$$\|S_f - u_f\|_2^2 \leq C\lambda\sqrt{|\triangle|}, \tag{10}$$

where C *is a positive constant independent of* $|\triangle|$, *the size of triangulation* $\triangle$.

Idea of Proof

Suppose u_f and S_f are the L^2 solution and the spline solution respectively. Project u_f to the spline space

$$u_f \stackrel{project}{\longrightarrow} \tilde{Q}u_f.$$

Then we have

$$E(u_f) \leq E(S_f) \leq E(\tilde{Q}u_f) \leq E(u_f) + err. \tag{11}$$

Functional difference

- Projection

$$u_f \xrightarrow{smooth} \eta_\delta * u_f = u_f^\delta \xrightarrow{project} Q\eta_\delta * u_f = Qu_f^\delta$$

- Error1

$$E(Qu_f^\delta) \leq E(u_f^\delta) + Err1$$

- Error2

$$E(u_f^\delta) \leq E(u_f) + Err2$$

Err2

- Lemma

  Suppose f is bounded. Then

  $$E(u_f^\delta) \leq E(u_f) + C\delta |u_f|_{\mathrm{BV}(\Omega)}$$

  where $C > 0$ is a constant dependent on $\|f\|_\infty$ and p.

- Proof.

$$\begin{aligned} & E(u_f^\delta) - E(u_f) \\ = & \left(J(u_f^\delta) - J(u_f)\right) \\ & + \frac{1}{2p\lambda}\left(\int_\Omega (\eta + |u_f^\delta - f|^2)^{p/2} - (\eta + |u_f - f|^2)^{p/2}\right) \end{aligned}$$

Lemma

Suppose f is bounded. Suppose that $\mathcal{S}$ contains a spline space $S_d^r(\triangle)$ for a degree $d \geq 3r+2$ as a subspace. Let $Qu_f^\delta \in \mathcal{S}$ be the quasi-interpolatory spline mentioned in Theorem 4. Then

$$E(Qu_f^\delta) \leq E(u_f^\delta) + C(\frac{|\Delta|}{\delta} + |\Delta|)|u_f|_{\mathrm{BV}(\Omega)}.$$

Err1: Variation term

$$
\begin{aligned}
&\left|\int_\Omega \sqrt{\epsilon + |\nabla Q u_f^\delta|^2} - \sqrt{\epsilon + |\nabla u_f^\delta|^2} dx\right| \\
&= \left|\int_\Omega \frac{|\nabla Q u_f^\delta|^2 - |\nabla u_f^\delta|^2}{\sqrt{1 + |\nabla Q u_f^\delta|^2} + \sqrt{\epsilon + |\nabla u_f^{\tilde{\delta}}|^2}} dx\right| \\
&\leq \int_\Omega \frac{|\nabla Q u_f^\delta - \nabla u_f^\delta||\nabla Q u_f^\delta + \nabla u_f^\delta|}{\sqrt{\epsilon + |\nabla Q u_f^\delta|^2} + \sqrt{\epsilon + |\nabla u_f^{\tilde{\delta}}|^2}} dx \\
&\leq \|\nabla (Q u_f^\delta - u_f^\delta)\|_{L^1}
\end{aligned}
$$

Theorem

Assume $d \geq 3r + 2$ and let $\triangle$ be a triangulation of Ω. Then there exists a quasi-interpolatory operator $Qf \in S_d^r(\triangle)$ mapping $f \in L_1(\Omega)$ into $S_d^r(\triangle)$ such that Qf achieves the optimal approximation order: if $f \in W^{m+1,p}(\Omega)$,

$$\|D_1^{\alpha} D_2^{\beta}(Qf - f)\|_{\Omega,p} \leq C|\triangle|^{m+1-\alpha-\beta}|f|_{\Omega,m+1,p} \qquad (12)$$

for all $\alpha + \beta \leq m + 1$ with $0 \leq m \leq d$. Here the constant C depends only on the degree d and the smallest angle $\theta_{\triangle}$ and may be dependent on the Lipschitz condition on the boundary of Ω.

Err1: Variation term

$$
\begin{aligned}
&\left|\int_\Omega \sqrt{\epsilon + |\nabla Q u_f^\delta|^2} - \sqrt{\epsilon + |\nabla u_f^\delta|^2} dx\right| \\
&\leq \|\nabla(Q u_f^\delta - u_f^\delta)\|_{L^1} \\
&\leq C|\Delta||u_f^\delta|_{W^{2,1}} \\
&\leq C\frac{|\Delta|}{\delta}|u_f|_{\mathrm{BV}(\Omega)}
\end{aligned}
$$

Err1: L^p term

$$
\begin{aligned}
&\left|\int_\Omega (\eta + (Qu_f^\delta - f)^2)^{p/2} - (\eta + (u_f^\delta - f)^2)^{p/2}\right| \\
&\leq \int_\Omega \left|\phi_\eta(\xi)(Qu_f^\delta - u_f^\delta)\right| \\
&\leq C \int_\Omega |(Qu_f^\delta - u_f^\delta)|\, dx \\
&\leq C|\Delta||u_f^\delta|_{W^{1,1}} \leq C|\Delta||u_f|_{\mathrm{BV}(\Omega)}
\end{aligned}
$$

Err: All together

$$E(u_f) \leq E(S_f) \leq E(Qu_f^\delta) \leq E(u_f) + C(\frac{|\Delta|}{\delta} + |\Delta| + \delta)|u_f|_{\mathrm{BV}(\Omega)}.$$

Let $\delta = \sqrt{\Delta}$.

Theorem

Fix $p \geq 1$ *and* $\eta > 0$*. Suppose* u_f *is the solution of problem (3) with* f *bounded. Then for any bounded function* u *with* $\sup|u| \leq \sup|f|$,

$$\|u - u_f\|_2^2 \leq C\lambda(E(u) - E(u_f))$$

for all $\epsilon \geq 0$*, where* C *depends on* $\sup|f|$ *and* η*.*

The difference of minimizers

Theorem

Fix $p \geq 1$ and $\eta > 0$. Suppose f is bounded. Suppose that $\mathcal{S}$ contains a spline space $S_d^r(\triangle)$ for a degree $d \geq 3r+2$ as a subspace. Let $\delta = \sqrt{|\triangle|}$. Then

$$E(S_f) - E(u_f) \leq C\sqrt{|\Delta|}\,E(0) \tag{13}$$

for all $\epsilon \geq 0$, where C is a constant dependent on f, p, and the smallest angle $\theta_\triangle$ of triangulation $\triangle$, and

$$\|S_f - u_f\|_2^2 \leq C\lambda\sqrt{|\triangle|}. \tag{14}$$

Algorithm

Given $u^{(k)}$, we find $u^{(k+1)} \in \mathcal{S}$ such that

$$\int_\Omega \frac{\nabla u^{(k+1)} \cdot \nabla \phi_j}{\sqrt{\epsilon + |\nabla u^{(k)}|^2}} dx + \frac{1}{\lambda} \int_\Omega \frac{u^{(k+1)} \phi_j}{(\eta + |u^{(k)} - f|^2)^{1-p/2}} dx$$
$$= \frac{1}{\lambda} \int_\Omega \frac{f \phi_j}{(\eta + |u^{(k)} - f|^2)^{1-p/2}} dx, \quad \forall j = 1, \cdots, n.$$

- TV- L^p recovery.

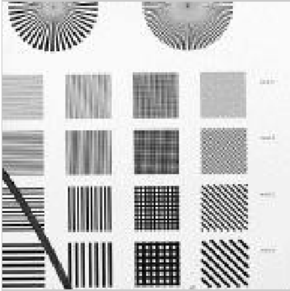

Figure: original image

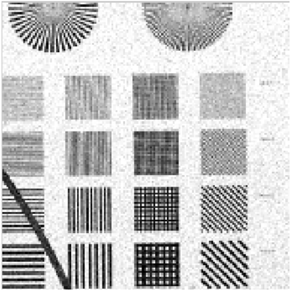

Figure: noisy image

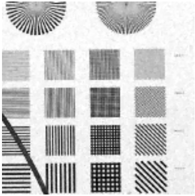

Figure: $L^{1.5}$ recovery

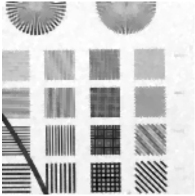

Figure: L^1 recovery

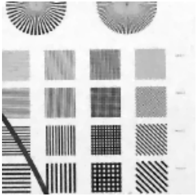

Figure: L^2 recovery

Experiments

- TV- L^p recovery.

Figure: original image

Figure: noisy image

Experiments

Figure: $L^{1.5}$ recovery

Figure: L^1 recovery

Figure: L^2 recovery

Experiments

	L_1	$L_{1.5}$	L_2
bike	29.73	30.21	30.14
lenna	21.91	30.98	27.79

Bibliography

- Allard, W.K., Total Variation Regularization for Image Denoising, III. Examples. SIAM Journal on Imaging Sciences, 2(2), 532-568, 2009
- Carasso, A., Singular integrals, image smoothness, and the recovery of texture in image deblurring. SIAM J. APPL. MATH. Vol. 64, No. 5, pp. 1749-1774.
- Chambolle, A. An algorithm for total variation minimization and applications. Journal of Mathematical Imaging and Vision, 20 (1-2): 89-97
- Chambolle, A.; Levine, S.; Lucier, B. ROF image smoothing: some computational comments. preprint
- Devore, R.; Lorentz, G. Constructive Approximation. Springer-Verlag. 1993.